Mathematical Foundations of Computer Graphics and Vision

Variational Methods III

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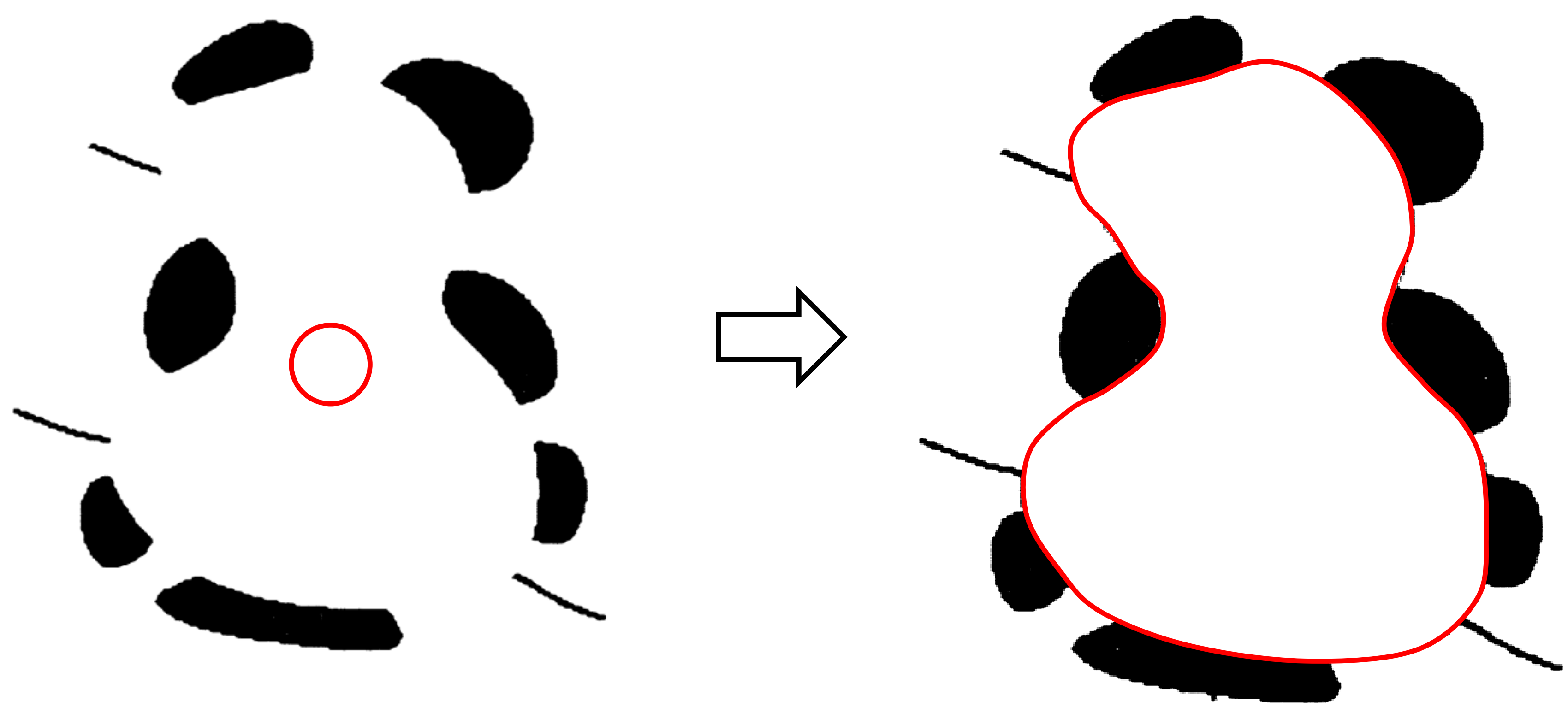
Problems

$$u^* = \underset{u \in \mathbb{C}^4([0,1],\mathbb{R}^2)}{\arg\min} \int_0^1 -E(u(s))^2 + \frac{\alpha}{2}\|\dot{u}(s)\|^2 + \frac{\beta}{2}\|\ddot{u}(s)\|^2 ds$$

$$\begin{aligned} \text{subject to} \quad & u(0) = u(1) \\ & \dot{u}(0) = \dot{u}(1) \\ & \ddot{u}(0) = \ddot{u}(1) \\ & \dddot{u}(0) = \dddot{u}(1) \\ & u \neq \emptyset \end{aligned}$$

- The global/local minimum of this functional is the **empty set** -> L(u) = 0
- Our gradient descent approach implicitly excluded it as a possible solution by just stopping to the first local minimum, but



The Balloon Term 

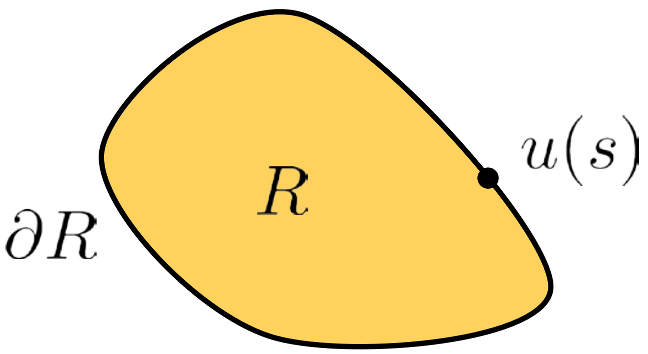
$$u^* = \arg\min_{u \in \mathbb{C}^4([0,1],\mathbb{R}^2)} \int_0^1 \dots ds + \underbrace{\gamma \int_{int(u)} dp}_{\text{Balloon term}}$$

- It measures the area inside the curve u (integral over all the internal points of u)
- $\gamma > 0$ penalizes big areas (force it too be small), $\gamma < 0$ penalizes small areas (force it too be big and also non-null)
- It is an integral over the interior points: Euler-Lagrange equation is not applicable

The Balloon Term 

The **Green's Theorem** (on the Cartesian plane)

$$\int_R (\nabla \times v)\, dp = \int_{\partial R} v \cdot dp$$



$$v : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

Vector field

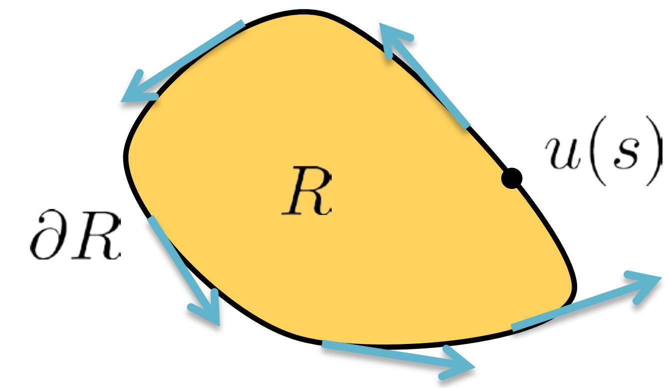
The curl of a vector field is a scalar field

$$\nabla \times v = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}$$

The Balloon Term 

The **Green's Theorem** (on the Cartesian plane)

$$\int_R (\nabla \times v)\, dp = \int_{\partial R} v \cdot dp$$



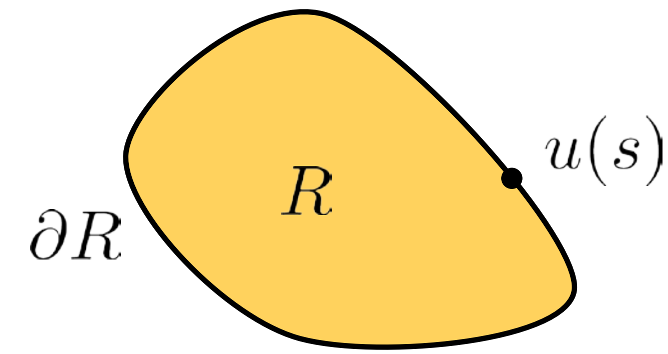
$v : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

Vector field

Oriented integration on an oriented curve (counterclockwise)

The Balloon Term 

The **Green's Theorem** (on the Cartesian plane)



$$v : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$\int_R (\nabla \times v)\, dp = \int_{\partial R} v \cdot dp$$

$$\int_R \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) dx dy = \int_{\partial R} v_x dx + v_y dy$$

$$= \int_0^1 v(u(s)) \cdot \dot{u}(s) \quad ds$$

$$\int_{int(u)} dp$$

$$\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = 1$$

The Balloon Term 

$$v(x,y) = \left(-\frac{y}{2}, \frac{x}{2}\right) \Rightarrow \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = 1$$

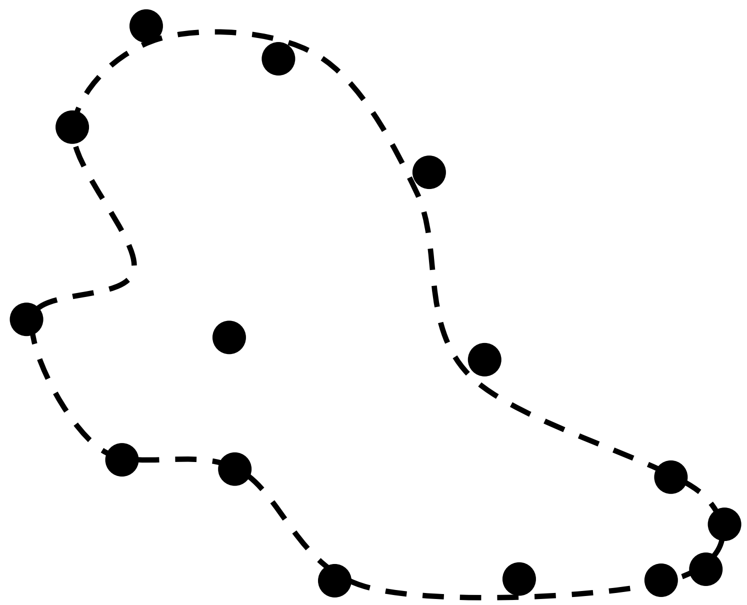
$$\int_{int(u)} dp = \int_0^1 v(u(s)) \cdot \dot{u}(s)\ ds = \frac{1}{2}\int_0^1 (-u_y(s), u_x(s)) \cdot \dot{u}(s)\ ds$$

Green's Theorem

Euler-Lagrange can be applied on this functional

Magically it will result in a force pushing the contour in/our along its normal

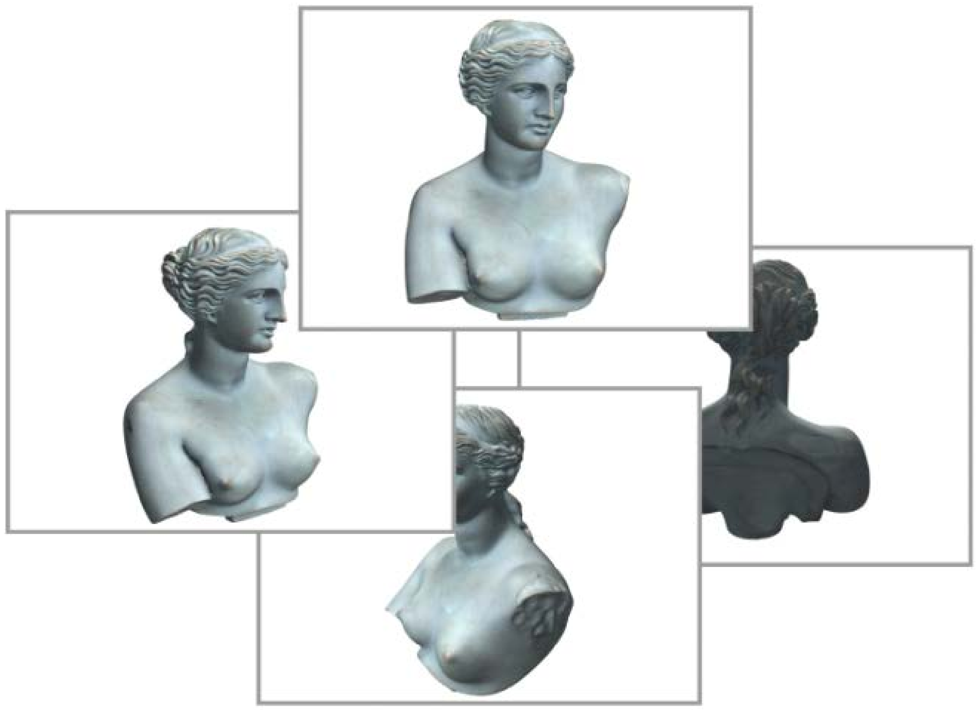
Curve Fitting



- In: Set of points $= \{q_i\}_i = \{(x_i, y_i)\}_i$

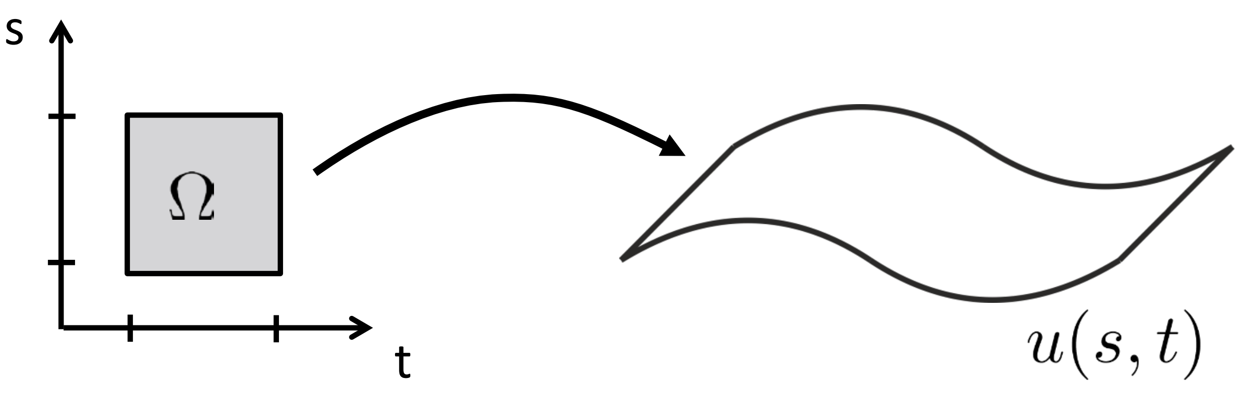
$$u^* = \underset{u \in \mathbb{C}^4([0,1],\mathbb{R}^2)}{\arg\min} \int_0^1 \frac{\alpha}{2}\|\dot{u}(s)\|^2 + \frac{\beta}{2}\|\ddot{u}(s)\|^2 ds$$

3D Surface Reconstruction

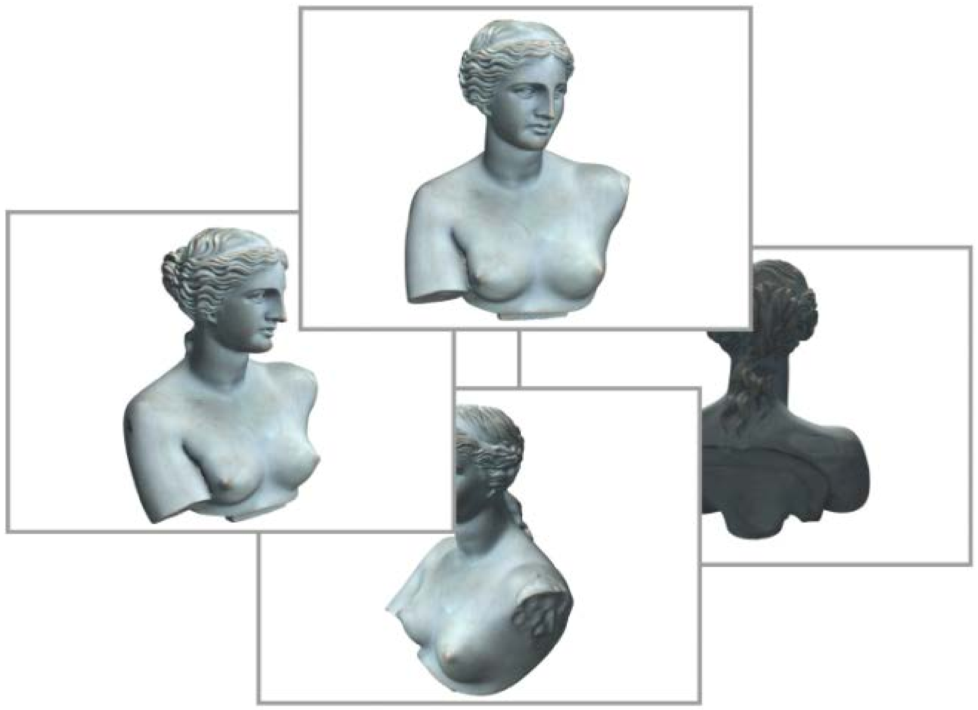


Input images

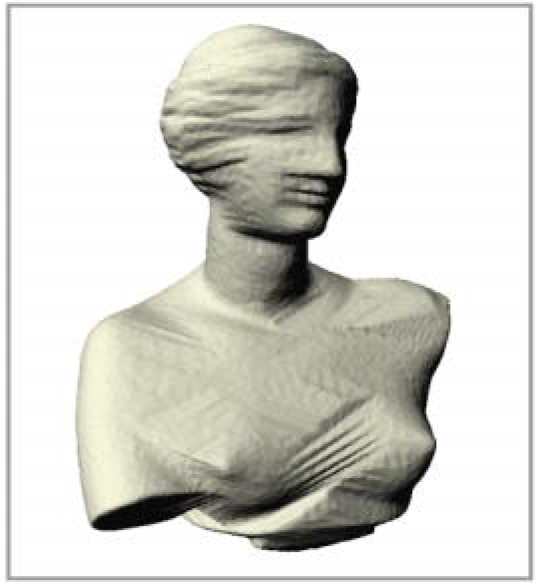
3D Surface



3D Surface Reconstruction

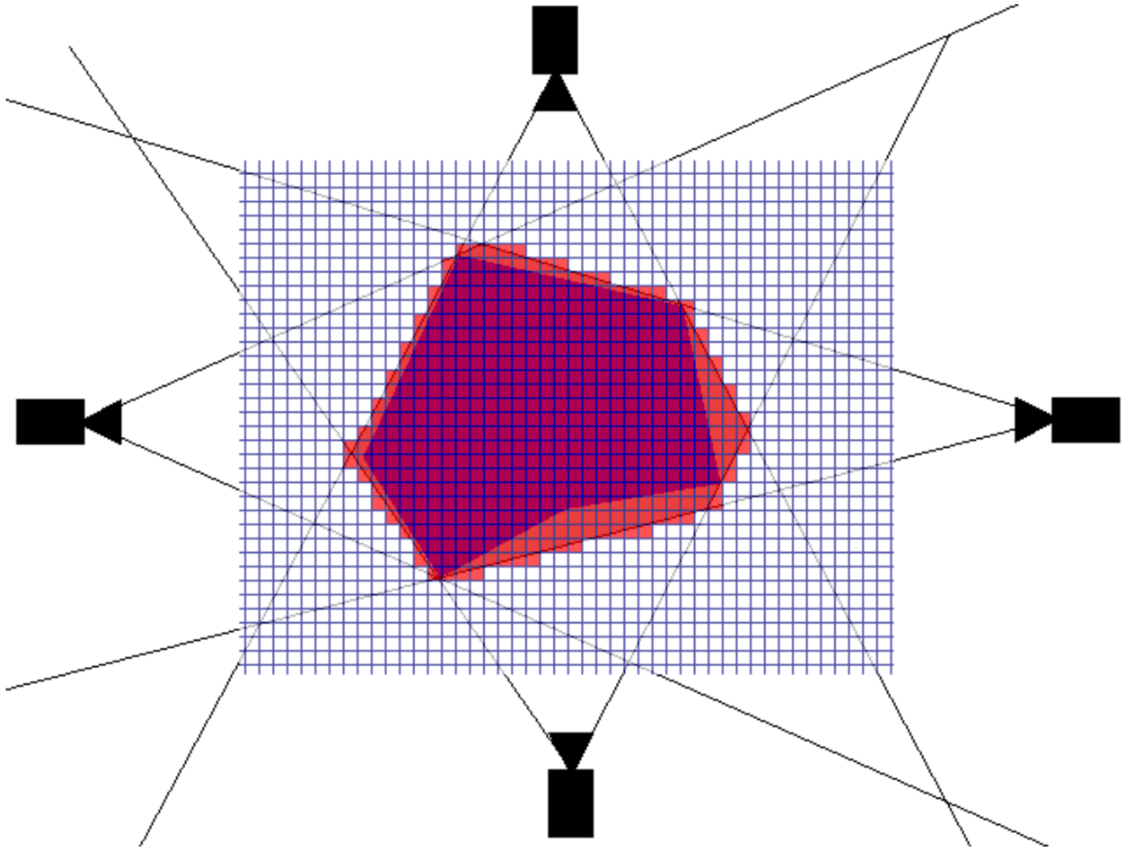


Input images

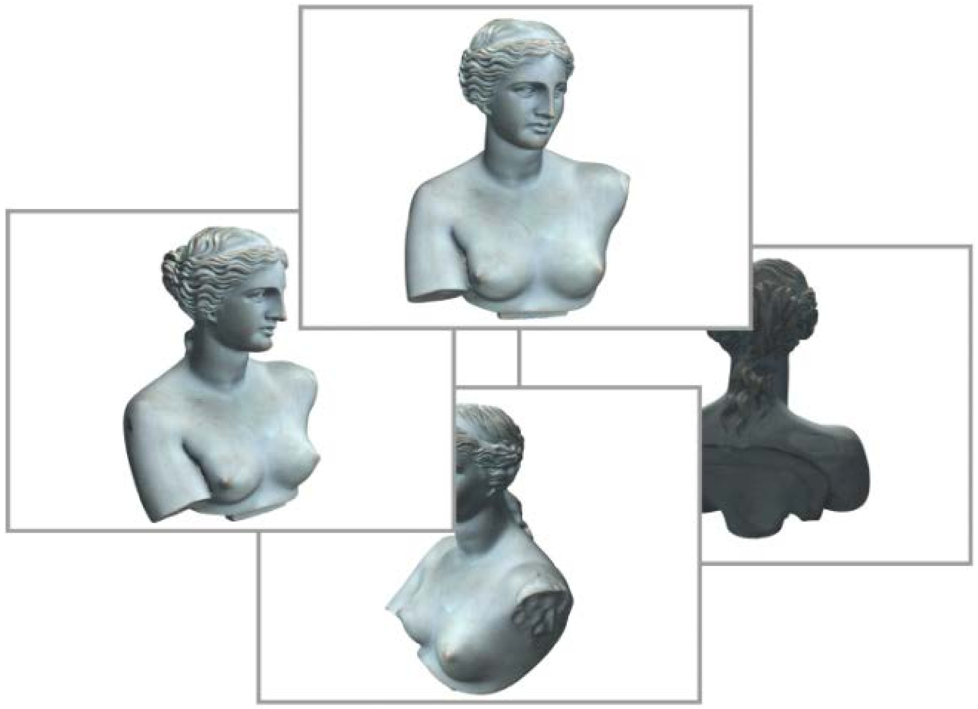


Shape from Silhouette
(visual hull)

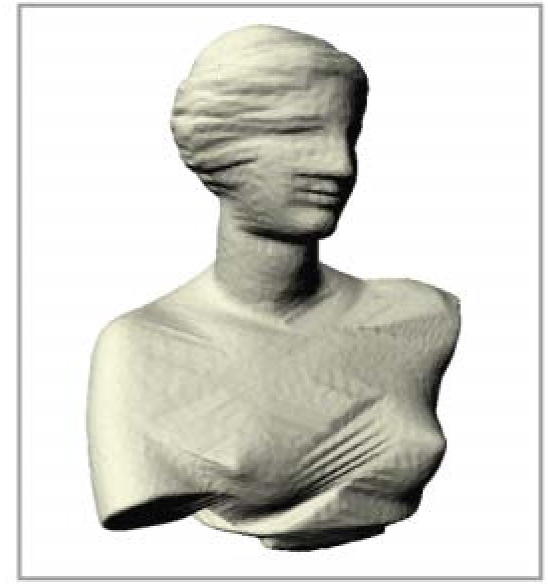
- Surface
- do not capture convexities



3D Surface Reconstruction



Input images



Shape from Silhouette
(visual hull)

- Surface
- do not capture convexities

Pairwise or Multi-view Stereo Matching

- Point cloud
- capture convexities

How do we fuse together the results of these two methods?

1st Approach

- Initial solution =  Visual Hull

- Optimize <u>locally</u>

$$\arg\min_{u\in\mathbb{C}^4(\Omega,\mathbb{R}^3)} \int_\Omega E(u(s,t))\,dsdt + \int_\Omega \left\|\frac{\partial u}{\partial s}\right\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 dsdt + \int_\Omega \left\|\frac{\partial^2 u}{\partial s^2}\right\|^2 + \left\|\frac{\partial^2 u}{\partial t^2}\right\|^2 + 2\left\|\frac{\partial^2 u}{\partial s\partial t}\right\|^2 dsdt$$

Penalizes surfaces far from the point cloud

 $\{q_i\}_i$

$$E(p) = \min\{\|p - q_i\|, \forall q_i\}$$

Membrane Energy penalizes non uniformly parameterized surfaces

Thin Plate Energy penalizes non smooth surfaces

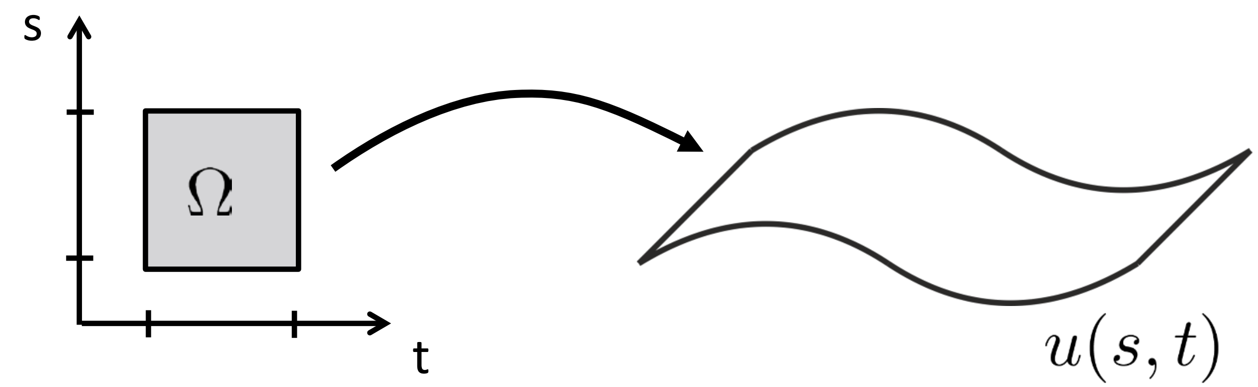
- Equal to the "total curvature"

$$\kappa = \kappa_1^2 + \kappa_2^2$$

iff the parameterization is uniform

Another Functional

- Given $L : C^4(\Omega \subseteq \mathbb{R}^2, \mathbb{R}^m) \to \mathbb{R}$



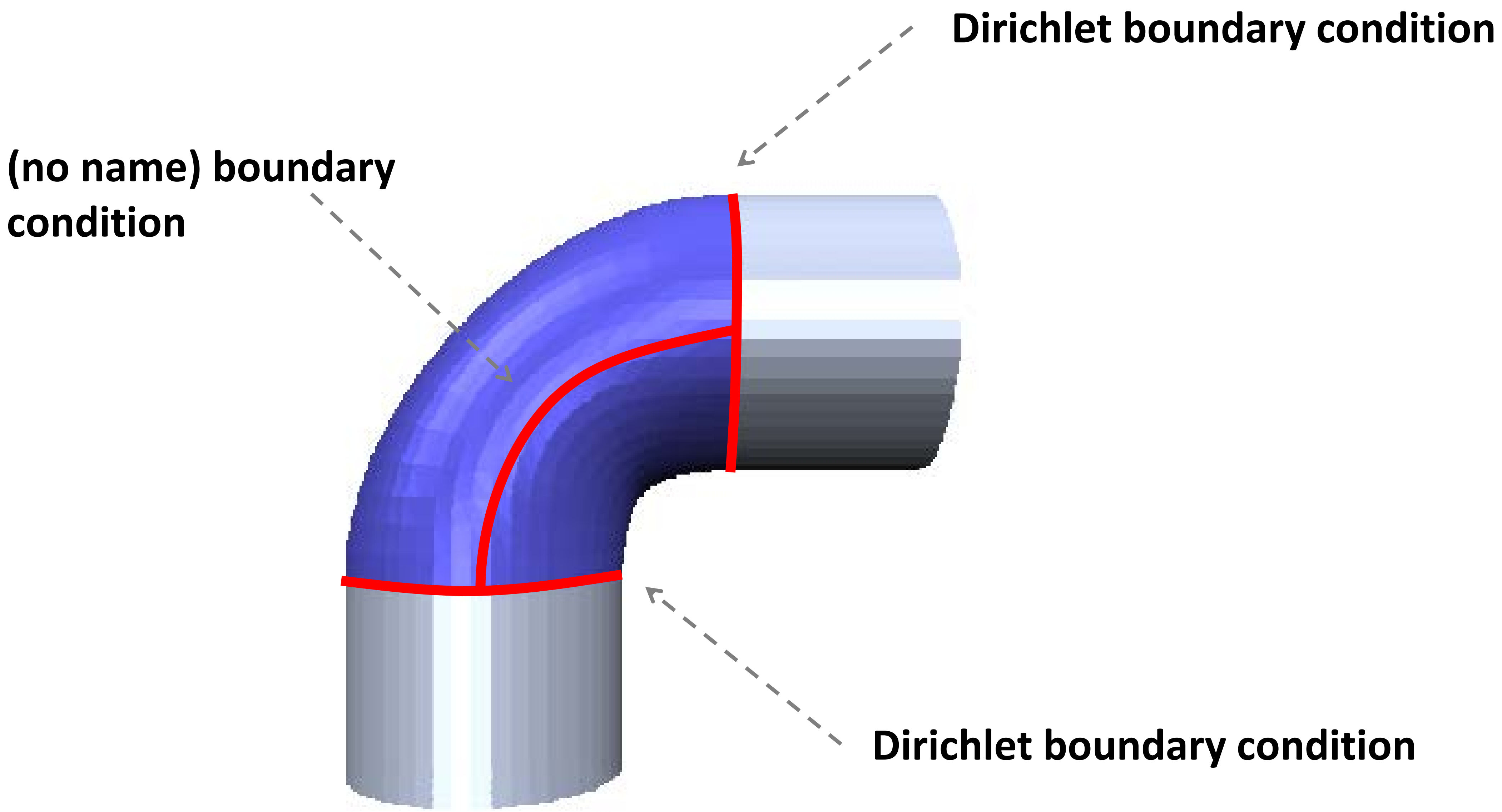
$$L(u) = \int_\Omega \psi\left(s, t, u, \frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial s^2}, \frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 u}{\partial s \partial t}, \frac{\partial^2 u}{\partial t \partial s}\right)\ ds dt$$

- The gradient in this case is

$$\nabla L(u) = \left(\frac{\partial \psi}{\partial u} - \frac{\partial}{\partial s}\frac{\partial \psi}{\partial \frac{\partial u}{\partial s}} - \frac{\partial}{\partial t}\frac{\partial \psi}{\partial \frac{\partial u}{\partial t}} + \frac{\partial^2}{\partial s^2}\frac{\partial \psi}{\partial \frac{\partial^2 u}{\partial s^2}} + \frac{\partial^2}{\partial t^2}\frac{\partial \psi}{\partial \frac{\partial^2 u}{\partial t^2}} + \frac{\partial^2}{\partial s \partial t}\frac{\partial \psi}{\partial \frac{\partial^2 u}{\partial s \partial t}} + \frac{\partial^2}{\partial t \partial s}\frac{\partial \psi}{\partial \frac{\partial^2 u}{\partial t \partial s}}\right)$$

- Boundary conditions?

Mixed Boundary Conditions



3D Surface Reconstruction

$$\underset{u \in \mathbb{C}^4(\Omega, \mathbb{R}^3)}{\arg\min} \int_\Omega E(u(s,t))\ dsdt + \int_\Omega \left\| \frac{\partial u}{\partial s} \right\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 dsdt + \int_\Omega \left\| \frac{\partial^2 u}{\partial s^2} \right\|^2 + \left\| \frac{\partial^2 u}{\partial t^2} \right\|^2 + 2 \left\| \frac{\partial^2 u}{\partial s \partial t} \right\|^2 dsdt$$

$$\nabla L(u) = \quad \nabla E(u(s,t))$$

?

$$\nabla L(u) = \left(\frac{\partial \psi}{\partial u} - \frac{\partial}{\partial s} \frac{\partial \psi}{\partial \frac{\partial u}{\partial s}} - \frac{\partial}{\partial t} \frac{\partial \psi}{\partial \frac{\partial u}{\partial t}} + \frac{\partial^2}{\partial s^2} \frac{\partial \psi}{\partial \frac{\partial^2 u}{\partial s^2}} + \frac{\partial^2}{\partial t^2} \frac{\partial \psi}{\partial \frac{\partial^2 u}{\partial t^2}} + \frac{\partial^2}{\partial s \partial t} \frac{\partial \psi}{\partial \frac{\partial^2 u}{\partial s \partial t}} + \frac{\partial^2}{\partial t \partial s} \frac{\partial \psi}{\partial \frac{\partial^2 u}{\partial t \partial s}} \right)$$

3D Surface Reconstruction

$$\underset{u \in \mathbb{C}^4(\Omega,\mathbb{R}^3)}{\arg\min} \int_\Omega E(u(s,t))\ dsdt + \int_\Omega \left\| \frac{\partial u}{\partial s} \right\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 dsdt + \int_\Omega \left\| \frac{\partial^2 u}{\partial s^2} \right\|^2 + \left\| \frac{\partial^2 u}{\partial t^2} \right\|^2 + 2 \left\| \frac{\partial^2 u}{\partial s \partial t} \right\|^2 dsdt$$

$$\nabla L(u) = \quad \nabla E(u(s,t)) \qquad -2\nabla^2 u \qquad 2\nabla^4 u$$

Discretizations:

$$\nabla = \left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right)$$

Gradient operator

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2}$$

Laplace operator $\xrightarrow{\hat{=}}$ **?**

$$\nabla^4 = \nabla^2\ \nabla^2 = \left(\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2} \right) \left(\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2} \right) = \frac{\partial^4}{\partial s^4} + \frac{\partial^4}{\partial t^4} + 2 \frac{\partial^4}{\partial s^2 \partial t^2}$$

Bi-Laplace operator $\xrightarrow{\hat{=}}$
(Biharmonic operator)

3D Surface Reconstruction

$$\underset{u \in \mathbb{C}^4(\Omega,\mathbb{R}^3)}{\arg\min} \int_\Omega E(u(s,t))\ dsdt + \int_\Omega \left\|\frac{\partial u}{\partial s}\right\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 dsdt + \int_\Omega \left\|\frac{\partial^2 u}{\partial s^2}\right\|^2 + \left\|\frac{\partial^2 u}{\partial t^2}\right\|^2 + 2\left\|\frac{\partial^2 u}{\partial s \partial t}\right\|^2 dsdt$$

$$\nabla L(u) = \quad \nabla E(u(s,t)) \qquad -2\nabla^2 u \qquad 2\nabla^4 u$$

Discretizations:

$$\nabla = \left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right)$$

Gradient operator

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2}$$

Laplace operator $\xrightarrow{\tilde{=}}$ $\tilde{\nabla}^2(v) = \frac{1}{\#N(v)}\left(\sum_{i \in N(v)} v_i\right) - v$

$$\nabla^4 = \nabla^2\ \nabla^2 = \left(\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2}\right) = \frac{\partial^4}{\partial s^4} + \frac{\partial^4}{\partial t^4} + 2\frac{\partial^4}{\partial s^2 \partial t^2}$$

Bi-Laplace operator (Biharmonic operator) $\xrightarrow{\tilde{=}}$ $\tilde{\nabla}^2(v) = \tilde{\nabla}\left(\tilde{\nabla}(v)\right)$

Observation

$$\underset{u \in \mathbb{C}^4(\Omega, \mathbb{R}^3)}{\arg\min} \int_\Omega E(\ldots t))\, dsdt + \int_\Omega \left\| \frac{\partial u}{\partial s} \right\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 dsdt + \int_\Omega \left\| \frac{\partial^2 u}{\partial s^2} \right\|^2 + \left\| \frac{\partial^2 u}{\partial t^2} \right\|^2 + 2 \left\| \frac{\partial^2 u}{\partial s \partial t} \right\|^2 dsdt$$

$$\nabla L(u) = \quad \nabla E(\ldots, t)) \qquad -2\nabla^2 u \qquad 2\nabla^4 u$$

Mesh smoothing

(filter the mesh using a gaussian filter)

[Taubin 95]

$$(\lambda + \mu)\nabla^2 u - (\lambda\mu)\nabla^4 u$$

- One step corresponds to a Gaussian filter pass (spectral analysis)
- The global minima is the empty set

Observation

$$\underset{u \in \mathbb{C}^4(\Omega,\mathbb{R}^3)}{\arg\min} \underbrace{\int_\Omega dsdt}_{} + \int_\Omega \left\| \frac{\partial u}{\partial s} \right\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 dsdt + \int_\Omega \left\| \frac{\partial^2 u}{\partial s^2} \right\|^2 + \left\| \frac{\partial^2 u}{\partial t^2} \right\|^2 + 2 \left\| \frac{\partial^2 u}{\partial s \partial t} \right\|^2 dsdt$$

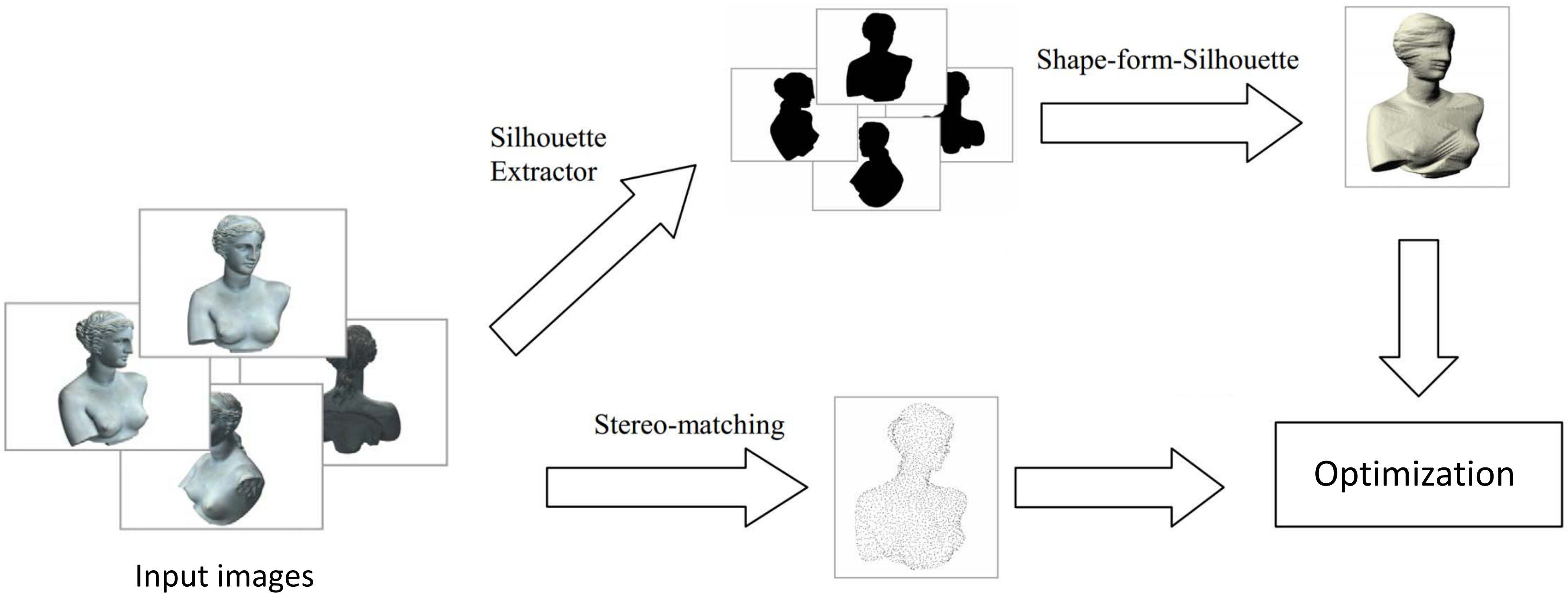
Balloon term

(volume preserving term)

[Taubin 95] -> Inflate term

- The global minima is a sphere with a similar volume

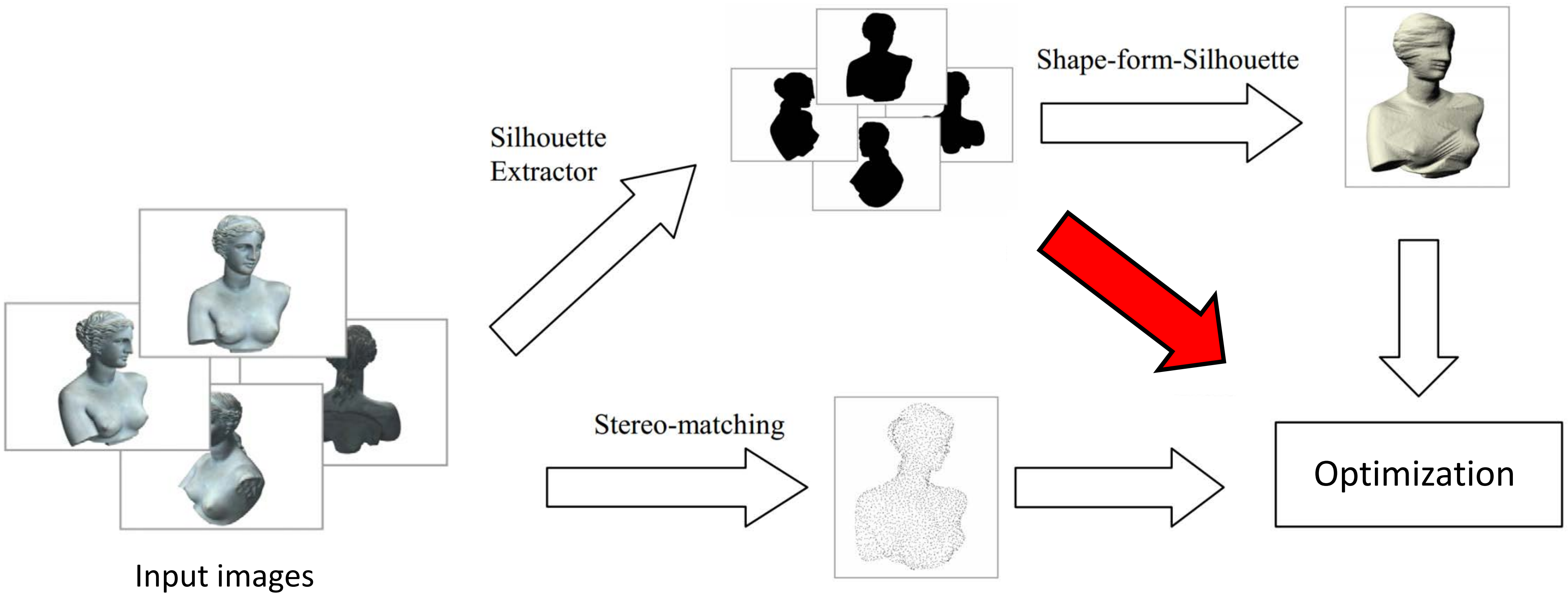
3D Surface Reconstruction



$$\arg\min_{u \in \mathbb{C}^4(\Omega,\mathbb{R}^3)} \int_\Omega E(u(s,t))\ dsdt + \int_\Omega \left\|\frac{\partial u}{\partial s}\right\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 dsdt + \int_\Omega \left\|\frac{\partial^2 u}{\partial s^2}\right\|^2 + \left\|\frac{\partial^2 u}{\partial t^2}\right\|^2 + 2\left\|\frac{\partial^2 u}{\partial s \partial t}\right\|^2 dsdt$$

- The local minima is a **smooth surface close to the point cloud** (computed by the stereo matching algorithm)
- If the visual hull is used to initialize the minimization, the resulting surface is also close to it

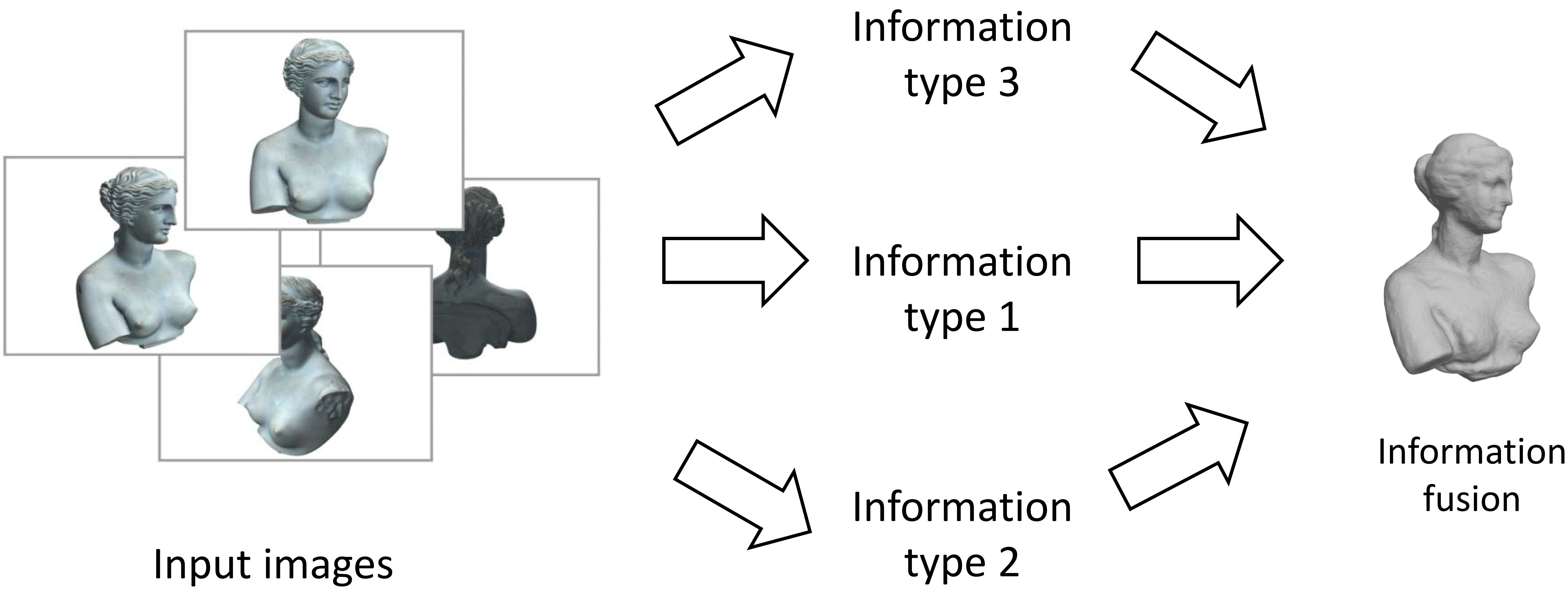
3D Surface Reconstruction



$$\arg\min_{u\in\mathbb{C}^4(\Omega,\mathbb{R}^3)} \int_\Omega E(u(s,t))\ dsdt + \int_\Omega \left\|\frac{\partial u}{\partial s}\right\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 dsdt + \int_\Omega \left\|\frac{\partial^2 u}{\partial s^2}\right\|^2 + \left\|\frac{\partial^2 u}{\partial t^2}\right\|^2 + 2\left\|\frac{\partial^2 u}{\partial s\partial t}\right\|^2 dsdt$$

$$+\int_\Omega D\left(\Pi\left(u\left(s,t\right)\right)\right) dsdt$$

Generalization



Images

An image can be viewed as a function

$$u : (\Omega \subseteq \mathbb{R}^n) \to \mathbb{R}^d$$

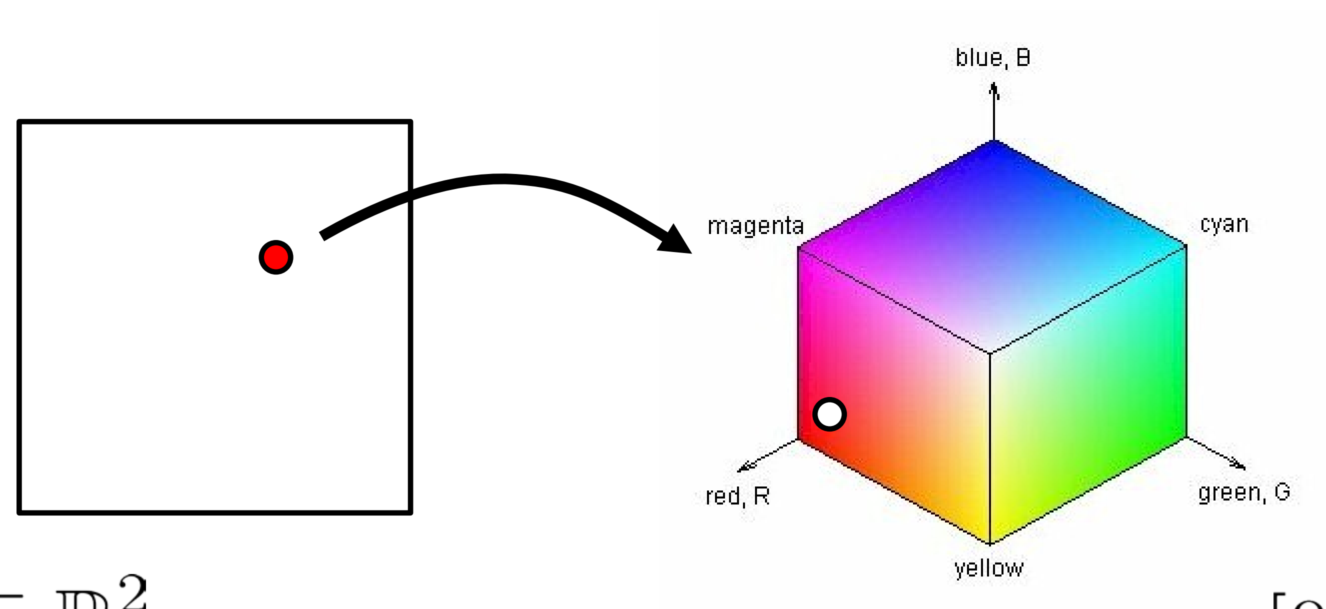
- n=2: Image
- n=3: Video (or a volumetric representation of a scene)
- n=4: Volumetric representation of a scene + time

- d=1: Brightness images/videos (or density volumes)
- d=3: Color images/videos/volumes
- d>1: Multispectral images/videos/volumes

Images

An image can be viewed as a function

$$u : (\Omega \subseteq \mathbb{R}^n) \rightarrow \mathbb{R}^d$$

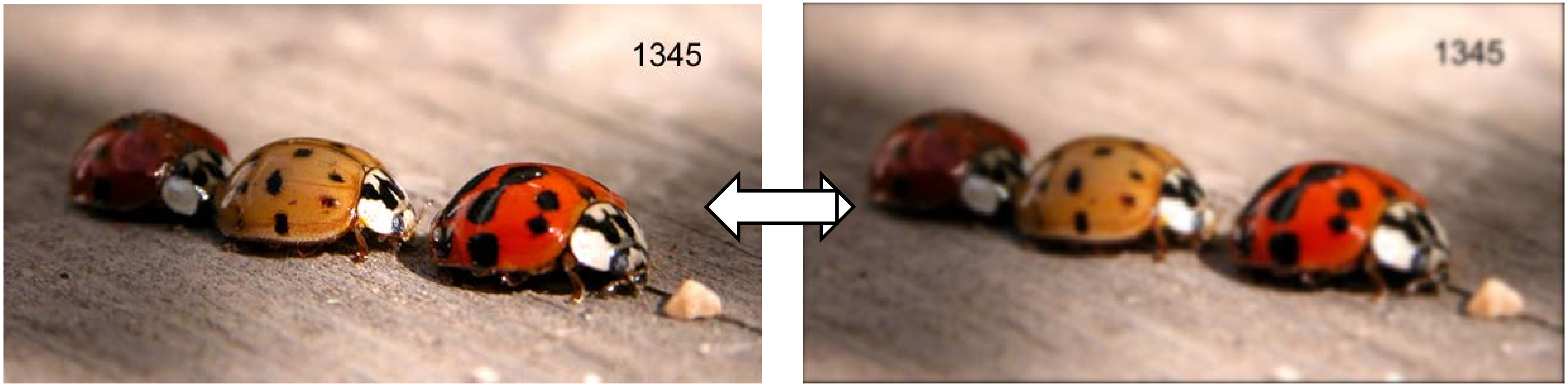


$\Omega \subseteq \mathbb{R}^2$

RGB cube

$[0, 1]^3 \subset \mathbb{R}^3$

Image De-convolution (De-blurring)



u $\qquad\qquad$ $f = A * u$

- **Image De-convolution:** Given an image f and a kernel A, we aim at recovering the image u such that $f = A * u$

- In a Variational framework, both u and f are modeled as continuous functions to the a color space

$$u : \Omega \rightarrow [0,1]^3$$
$$f : \Omega \rightarrow [0,1]^3$$

← RGB or YCbCr/YUV
(channels are de-correlated)

Image De-convolution (De-blurring)

u

$f = A * u$

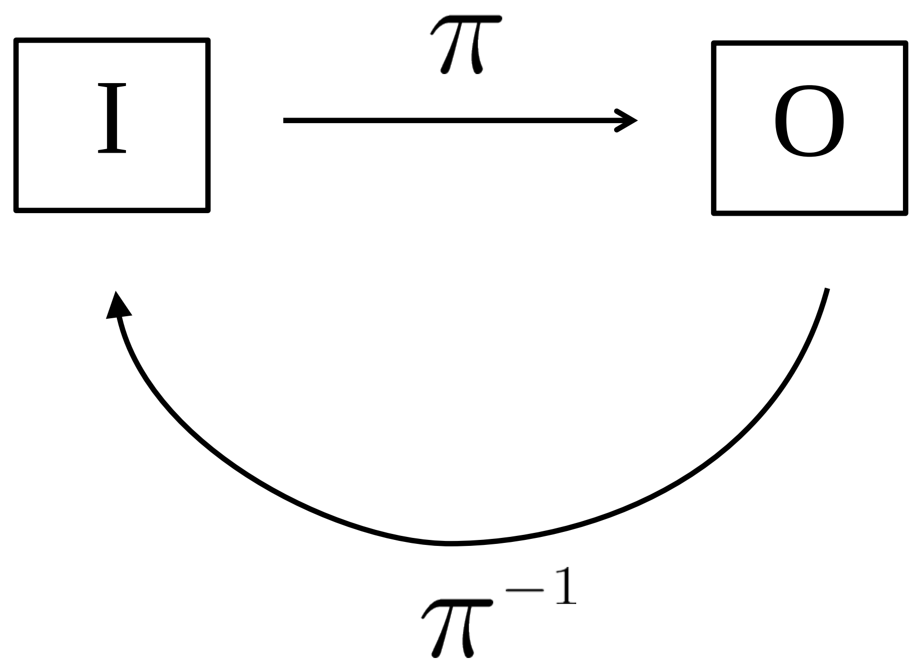
- find u such that $f = A * u$

$$\Downarrow$$

$$\arg\min_{u} \int_{\Omega} \|A * u - f\|^2$$

Generative approach to the problem
(problem solving paradigm)

Generative Approach



Formal definition of a **problem** π

i = problem instance
$\pi(i)$ = solution of the problem instance i

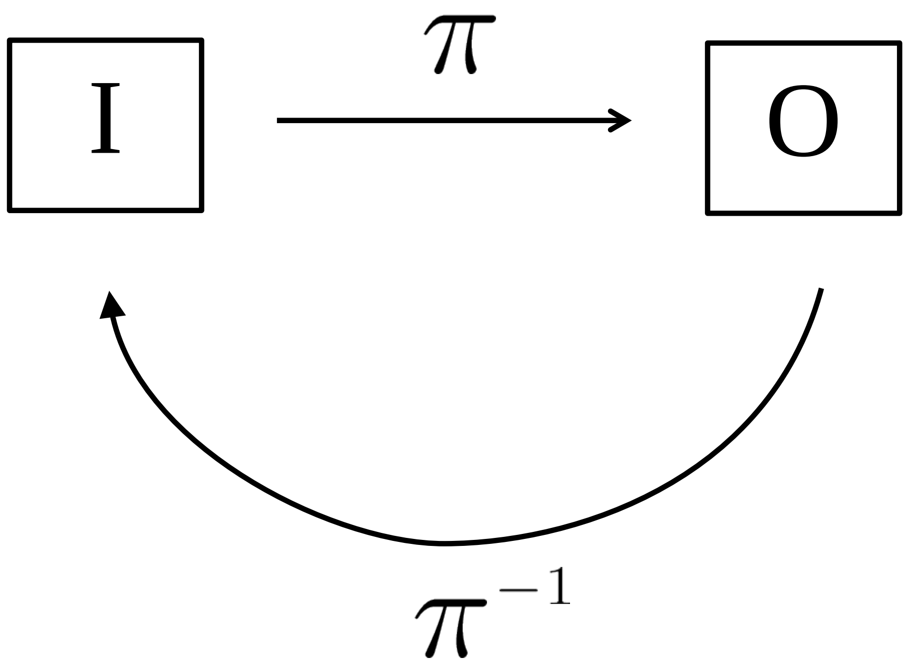
$\pi(i)$ is typically difficult to compute

$$\pi(i) = \arg\min_{o} \mathbf{L}\left(\pi^{-1}(o), i\right)$$

Generative approach to the problem

- **L** is a loss functional evaluating two input elements (e.g. a distance)

Generative Approach



Formal definition of a **problem** π

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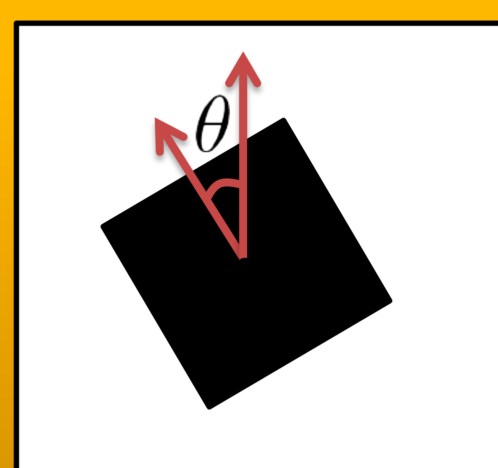
- **Example:**

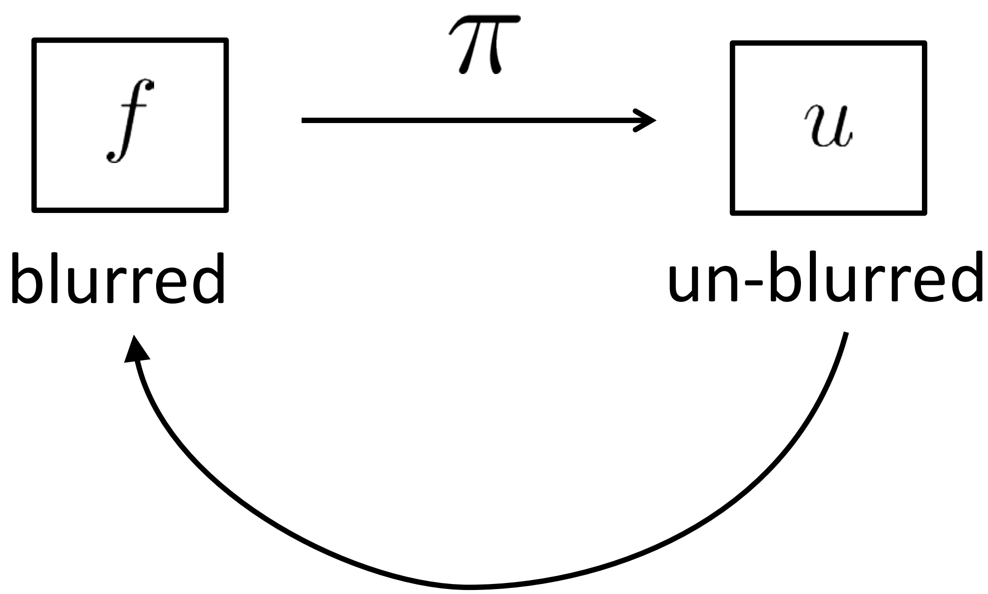
I =



Image

$\xrightarrow{\pi}$ O = orientation angle θ of the square

Generative Approach: Image De-blurring



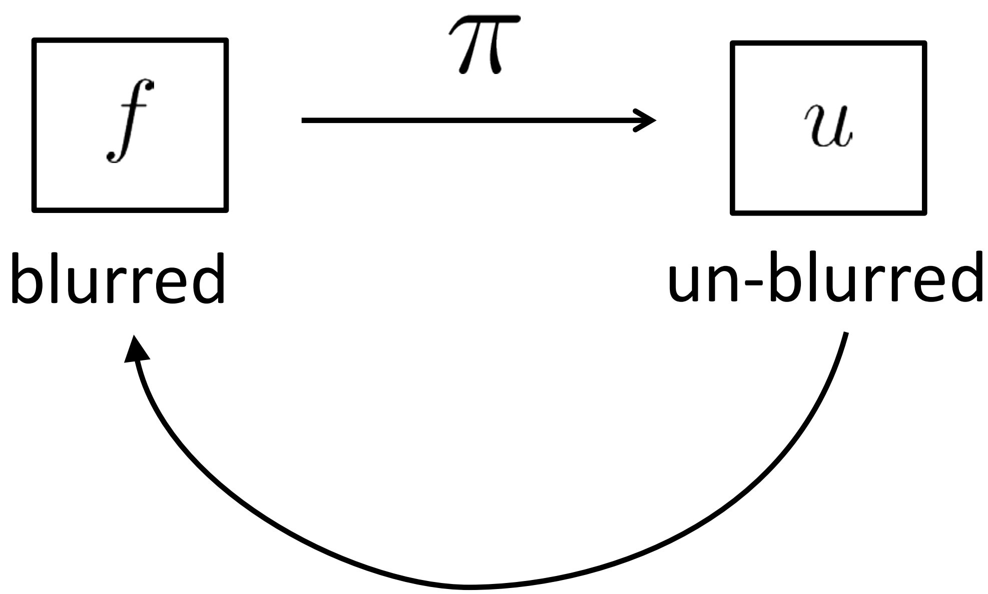
$A * u$ very easy to compute

$$\pi(f) = \arg\min_u \mathbf{L}\left(A * u, f\right)$$

Generative approach to the problem

- test (all) the u
- convolve them with A
- evaluate a cost functional with the original f

Generative Approach: Image De-blurring



$A * u$ very easy to compute

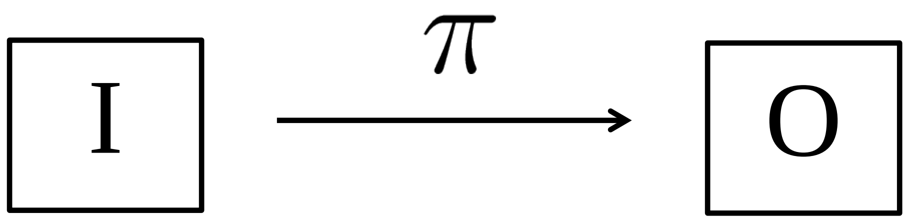
$$\pi(f) = \arg\min_u \mathbf{L}\left(A * u, f\right)$$

Generative approach to the problem

$$= \arg\min_u \int_\Omega \|(A * u)(p) - f(p)\|^2 dp$$

- What does the choice of the loss functional corresponds to?
- Is there any meaning?
- Can we choose any arbitrary functional?

Generative Approach seen as a MAP or ML



$$\pi(i) = \arg\max_o P(O = o | I = i)$$

Maximum a posteriori estimator
(the desired o is the one with maximum probability assuming the input i)

*

$$= \arg\min_o -\log\left[P\left(I = i | O = o\right)\right] - \log\left[P\left(O = o\right)\right]$$

uniform

Likelihood
(loss functional **L**)

Prior on o
(additional information about our output)

$$= \arg\max_o P\left(I = i | O = o\right)$$

Maximum Likelihood
(the desired o is the one which generates the observed input i with the maximum probability, i.e., which is likely to generate i.)

Image De-convolution (De-blurring)

$$\arg\min_u \int_\Omega \|(A * u)(p) - f(p)\|^2 dp$$

Generative approach to the problem
(the two quantities should be equal up to
Gaussian noise)

*

*

$\Updownarrow$

$$\arg\min_u \|A * u - f\|_2^2$$

norm L2

*

these two objects has to be close

Image De-convolution (De-blurring)

$$\arg\min_u \int_\Omega \|(A * u)(p) - f(p)\|^2 dp + \int_\Omega \|\nabla u(p)\|^2 dp$$

⇕

$$\arg\min_u \|A * u - f\|_2^2 + \|\nabla u\|_2^2$$

Prior on u

Image De-convolution (De-blurring)

$$\arg\min_u \int_\Omega \|(A*u)(p) - f(p)\|^2 dp + \int_\Omega \|\nabla u(p)\| dp$$

⇕

$$\arg\min_u \|A*u - f\|_2^2 + \|\nabla u\|_1$$

Prior on u

*

L1 Prior + L2 "linear" cost = **Lasso problem**

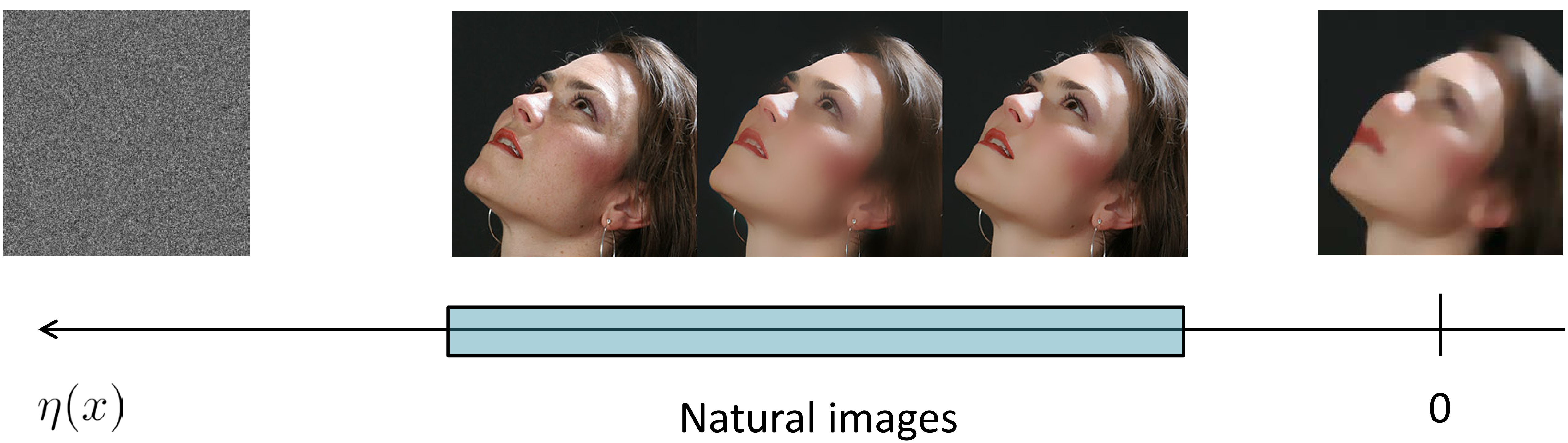
⇓

$$\nabla u(x) = \sum_i \delta(x - p_i) + \eta(x)$$

Sparse gradient

Natural images and Sparsity

$$\nabla u(x) = \sum_i \delta(x - p_i) + \eta(x)$$



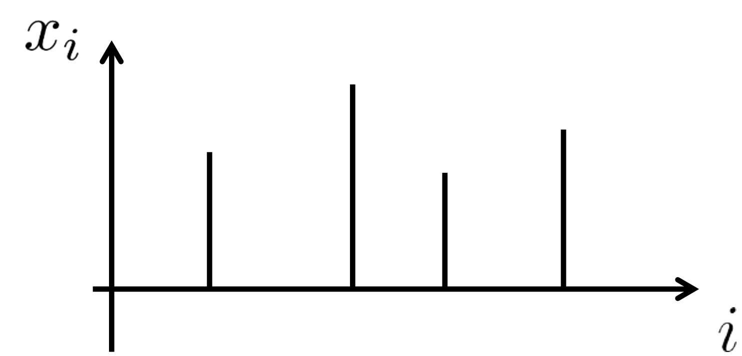
Lasso Problem

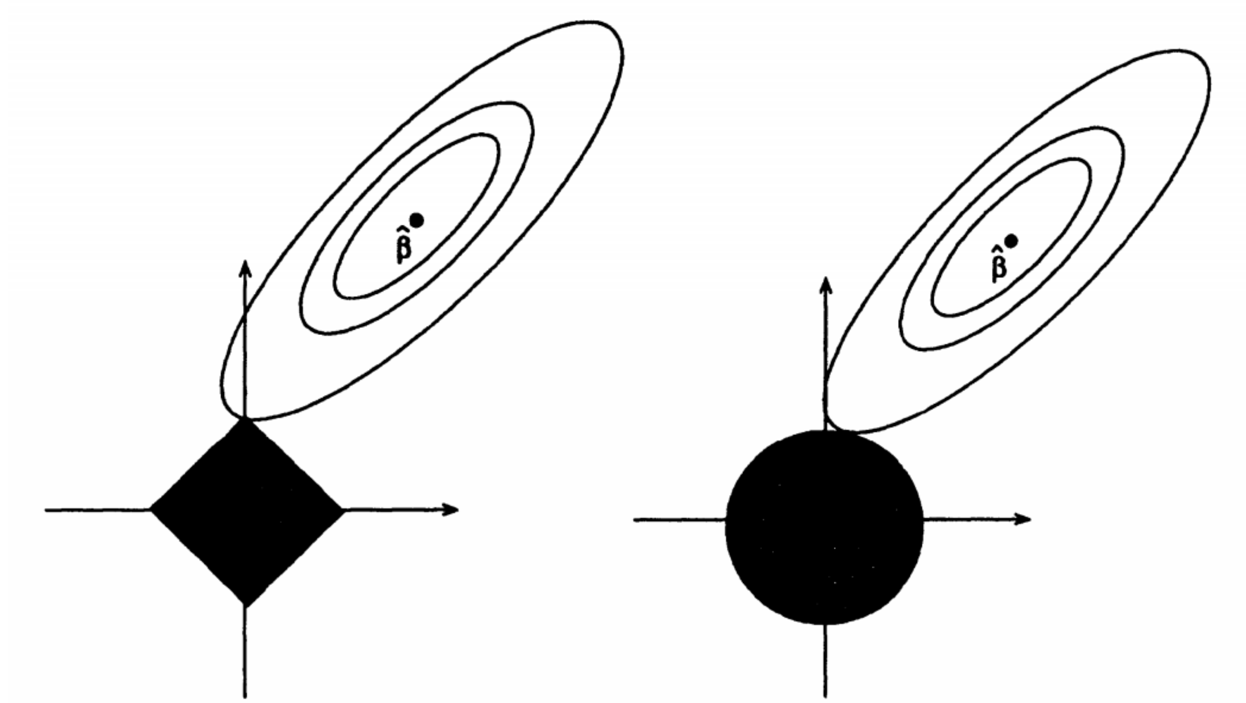
$$\arg\min_{x} \quad \|L(x) - y\|_2^2 + \|x\|_1$$

Tibshirani, R., "Regression shrinkage and selection via the lasso". 1996
Journal of the Royal Statistical Society
(useful in Compressed Sensing)

$L(\cdot)$ is linear and orthonormal $\Rightarrow$ Solution will be sparse

$$\begin{cases} L(x) = Ax \\ AA^T = I \end{cases}$$





Lasso Problem

$$\arg\min_x \quad \|L(x) - y\|_2^2 + \|x\|_1$$

Tibshirani, R., "Regression shrinkage and selection via the lasso". 1996
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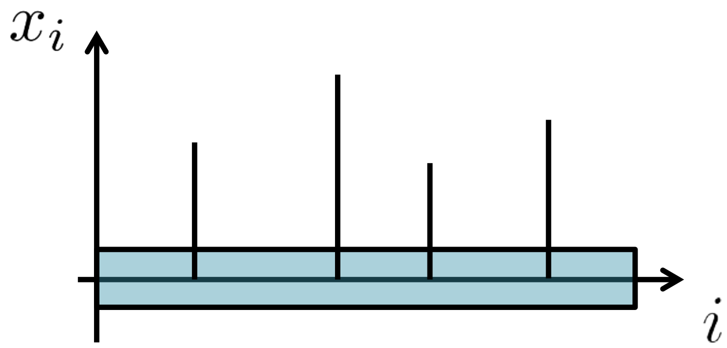
$L(\cdot)$ is linear $\Rightarrow$ Solution will **typically** be sparse

The further one goes from the linearity and ortho-normality of $L(\cdot)$ $\Rightarrow$ The more the sparse property will disappear



Infinite Dimensional Case

$$\begin{cases} \arg\min_u \quad \|L(u) - y\|_2^2 + \|\nabla u\|_1 \\ L(\cdot) \quad \text{is linear and orthonormal} \end{cases} \Rightarrow \quad \nabla u \quad \text{will be sparse}$$

Proof:

$$\begin{cases} g = \nabla u \\ I(g)(q) = \int_{\gamma[p,q]} g(r) \cdot dr \end{cases} \Rightarrow \quad I(\nabla u)(q) = u(q) - u(p)$$

$$\int_{\gamma[p,q]} \nabla u(r) \cdot dr = u(q) - u(p)$$ Gradient theorem

Infinite Dimensional Case

$$\begin{cases} \arg\min_u \quad \|L(u) - y\|_2^2 + \|\nabla u\|_1 \\ L(\cdot) \quad \text{is linear and orthonormal} \end{cases} \Longrightarrow \quad \nabla u \quad \text{will be sparse}$$

Does it work in infinite dimensional case???

Proof:

$$\begin{cases} g = \nabla u \\ I(g)(q) = \int_{\gamma[p,q]} g(r) \cdot dr \end{cases} \Longrightarrow \quad I(\nabla u)(q) = u(q) - u(p) \quad = u(q)$$

Linear operator (orthonormal?????)

select p in such a way $u(p) = 0$ (maybe in a restricted domain of u)

$$= \|L(I(g)) - y\|_2^2 + \|g\|_1$$

Combination of linear operators is linear

C.V.D.

Image De-convolution (De-blurring)

$$\arg\min_u \int_\Omega \|(A * u)(p) - f(p)\|^2 dp + \int_\Omega \|\nabla u(p)\| dp$$

$$\Updownarrow$$

$$\arg\min_u \|A * u - f\|_2^2 + \|\nabla u\|_1$$

Lasso problem

Convolution is a linear operator

$\|\nabla u\|_1$ Total Variation (TV)

$\arg\min_u \|A * u - f\|_2^2 + \|\nabla u\|_1$ TV-L2 = L2 cost + Total Variation

Image De-convolution (De-blurring)

$$\arg\min_u \int_\Omega \|(A * u)(p) - f(p)\|^2 dp + \int_\Omega \|\nabla u(p)\| dp$$

Not derivable in 0 ⇨ it is not C^2

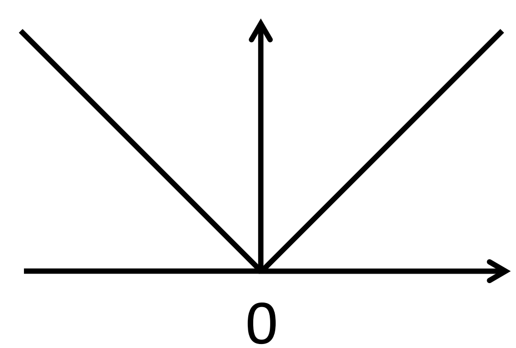
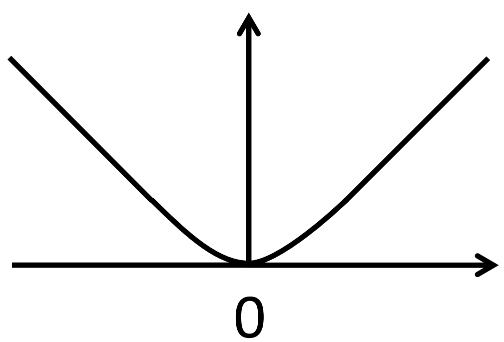


Image De-convolution (De-blurring)

$$\arg\min_u \int_\Omega \|(A * u)(p) - f(p)\|^2 dp + \int_\Omega \|\nabla u(p)\|_\epsilon dp$$

Norm-1 (ϵ): C^2 approximation $\|x\|_\epsilon = \sqrt{\sum x_i^2 + \epsilon}$



$$\frac{\partial}{\partial x}\|x\|_\epsilon = \frac{x}{\|x\|_\epsilon}$$

$$\nabla L(u) = 2A * (A * u - f) - div\left(\frac{\nabla u}{\|\nabla u\|_\epsilon}\right)$$

Gradient of our functional

(only if the kernel A is symmetric)

Image De-convolution (De-blurring)

```
Filter = fspecial('gaussian', 30,2);
f=imfilter(I,Filter);

u=f;
a=0.0001;
b=1;
epsilon=0.1;
it=400;

for o=1:it
    fu=imfilter(u,Filter) - f;
    fu=-imfilter(fu,Filter);

    [ux,uy]=gradient(u);
    norm=sqrt(ux.*ux+uy.*uy+epsilon);
    ux=ux./norm;
    uy=uy./norm;

    [uxx]=gradient(ux);
    [uxy,uyy]=gradient(uy);
    DIV=uxx+uyy;

    u = u + b*(fu+a*DIV);
end
```

Image De-noising

u

$f = u + \mu$

$$\arg\min_u \ \|u - f\|_2^2 + \|\nabla u\|_1$$

Generative approach to the problem

*

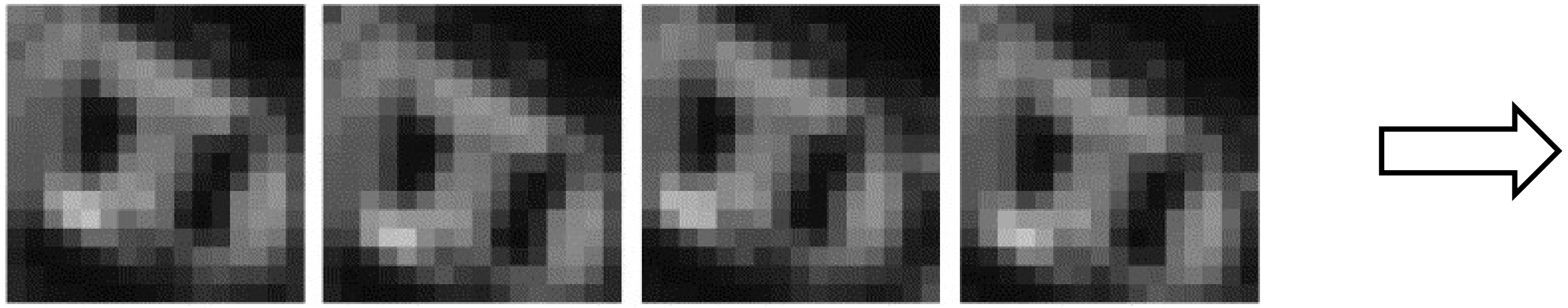
L2 + L1 = Lasso problem

L2 + TV = TV-L2 problem

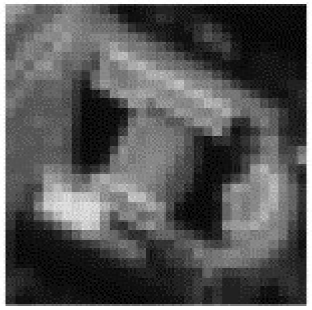
$$\arg\min_u \ \|u - f\|_1 + \|\nabla u\|_1$$

L1 + TV = TV-L1 problem (No lasso)

Video Super-Resolution



sequence of images
(of the same scene with the camera translating a bit)



super resolution result
(aerial image of a building)

- **Why it works?** a sequence of n images provides n observations of a point in the scene: sometimes in the center, sometimes ¼ of a pixel on the right, sometimes on the left, top, down etc…

- We just need to fuse all these information together.

- **Secondary objectives**: - Eliminate sensor noise from the video (thermal noise or spikes (video restoration))
  - Eliminate not wanted occlusions

Video Super-Resolution

- **Video Super-Resolution:** given a sequence of images f_i representing the same image u translated by w_i and down-sampled, find the original image u

$$u(x, y)$$

Original image (our unknown)

$$u_{w_i}(x, y) = u(x + w_i^x, y + w_i^y)$$

u translated by $w_i = (w_i^x, w_i^y)$
(sub-pixel accuracy needed)

$$A * u_{w_i}$$

Down-Sampling
(modeled as convolution with a sinc kernel,
i.e., a low pass filter)

$$\Downarrow$$

$$\sum_{i=1}^{n} \int_{\Omega} \|(A * u_{w_i})(p) - f_i(p)\|^2 dp$$

Generative approach

Video Super-Resolution

- **Video Super-Resolution:** given a sequence of images f_i representing the same image u translated by w_i and down-sampled, find the original image u

$u(x, y)$ — Original image (our unknown)

$u_{w_i}(x, y) = u(x + w_i^x, y + w_i^y)$ — u translated by $w_i = (w_i^x, w_i^y)$ (sub-pixel accuracy needed)

$A * u_{w_i}$ — Down-Sampling (modeled as convolution with a sinc kernel, i.e., a low pass filter)

$\Downarrow$

$$\sum_{i=1}^{n} \|A * u_{w_i} - f_i\|_2^2$$

Generative approach

Video Super-Resolution

$$\sum_{i=1}^{n} \|A * u_{w_i} - f_i\|_2^2 + \|\nabla u\|_1$$

← L1 Prior + L2 cost = Lasso

⇓

Solution is piecewise smooth

↗

Sum of L2-Norm is still a 2-type-Norm

⇓

The two quantities should be equal
up to a Gaussian noise

Video Super-Resolution

$$\sum_{i=1}^{n} \|A * u_{w_i} - f_i\|_2^2 + \|\nabla u\|_1$$

What about an L1 cost, instead?

$$\sum_{i=1}^{n} \|A * u_{w_i} - f_i\| + \|\nabla u\|_1$$

L1 Prior + L1 cost = Lasso

(not guarrantee to have piecewise smooth solutions but it behaves similarly due to the median theorem (see later)

Sum of L1-Norm is a 1-type-Norm

The two quantities should be equal up to a Gaussian noise with spikes

More robust to burst noise/spikes/outliers

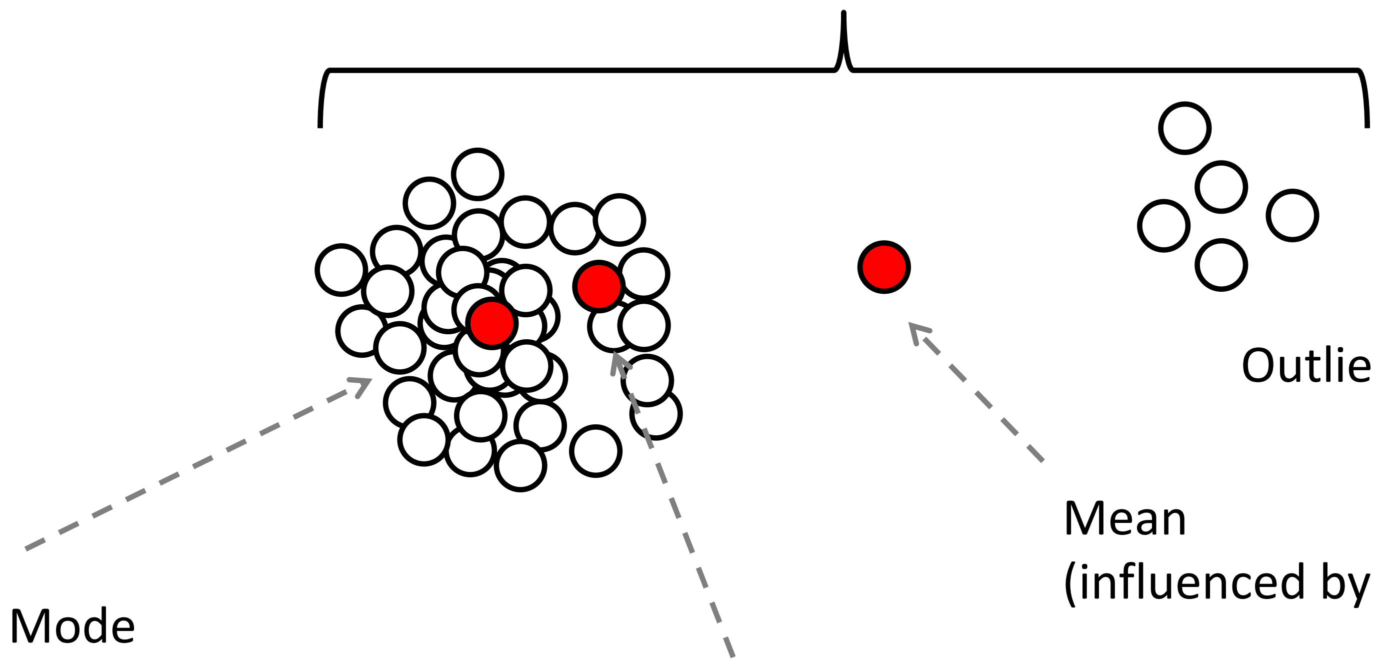
Why?

$$\sum_{i=1}^{n} \|A * u_{w_i} - f_i\|_2^2$$

When you have multiple information regarding a single unknown

The result of the minimization depends on the norm

The p-type-Norm Minimizations



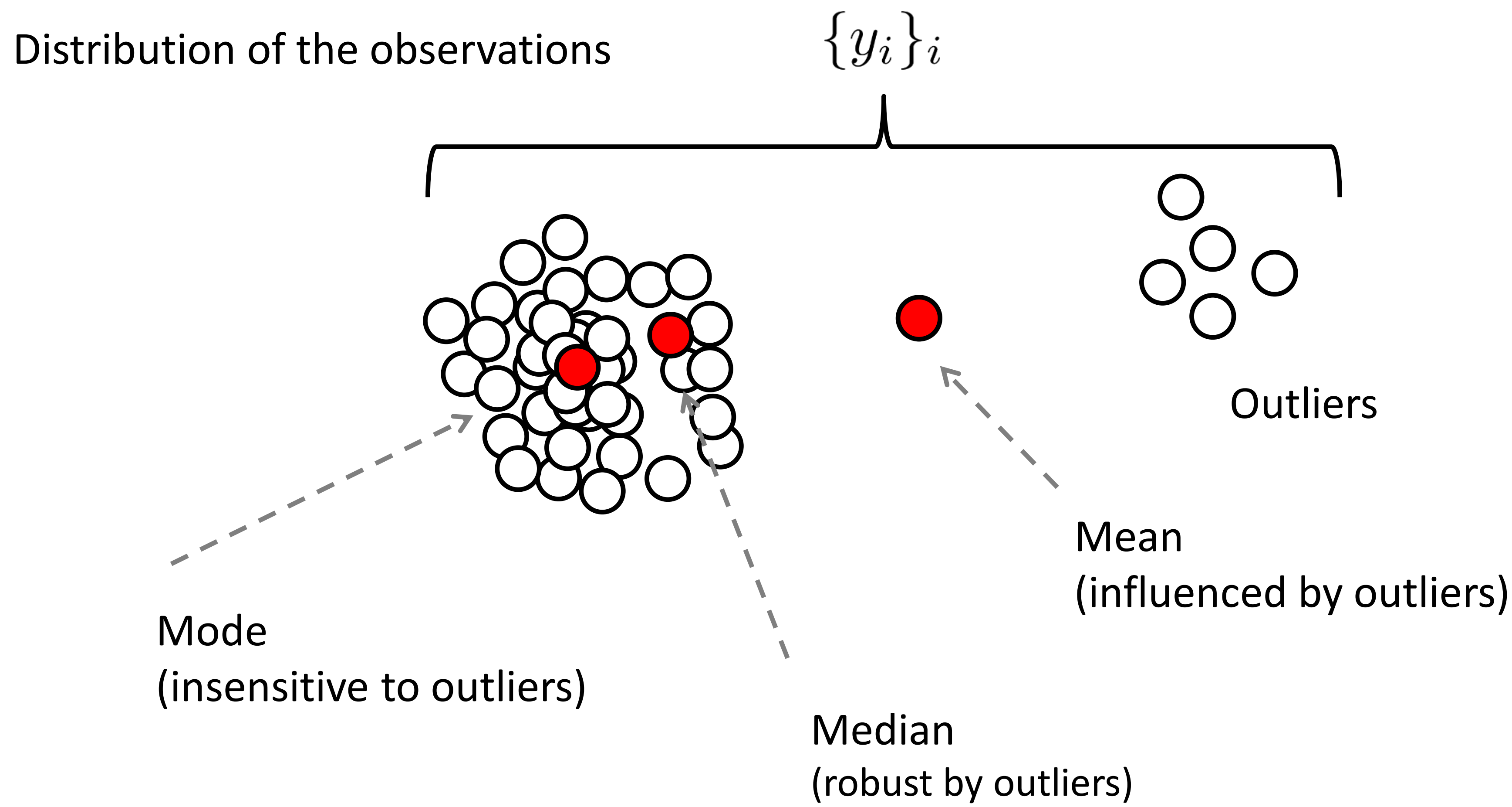
$$\arg\min_x \sum_i \|x - y_i\|_2$$ Mean

$$\|x - y\|_\epsilon$$

$$\arg\min_x \sum_i \|x - y_i\|_1$$ Median ← Not differentiable, difficult to optimize

$$\arg\min_x \sum_i \|x - y_i\|_0$$ Mode

The p-type-Norm Minimizations



$$\arg\min_x \sum_i \|x - y_i\|_2$$ Mean

$$\arg\min_x \sum_i \|x - y_i\|_1$$ Median

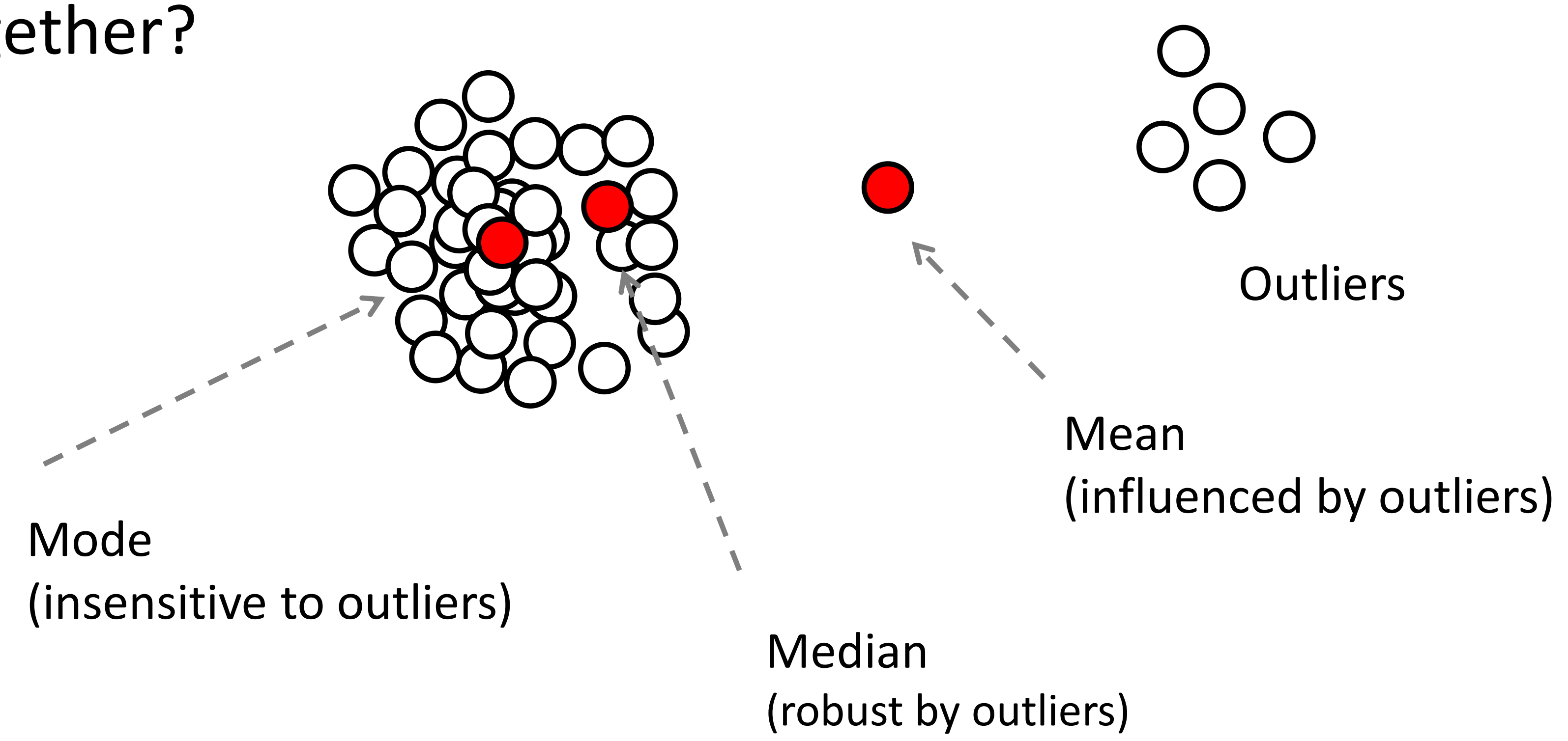
$$\|x - y\|_{0.001,\epsilon}$$

$$\arg\min_x \sum_i \|x - y_i\|_0$$ Mode

← Not a norm, difficult to optimize

Conclusions: in general…

- if one has multiple information regarding an unknown, how does he fuse them together?



- so.. the best way is to sum them together and minimize a functional

$$\arg\min_x \sum_i \|x - y_i\|_p$$

Does the used norm matter?