
Rigid Transformations
--- the geometry of SO(3) & SE(3) ---

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Motivation

\[ x^* = \arg \min_{x \in \mathbb{R}^k} L(x) \]  
(unconstrained minimization problem)

\[ u^* = \arg \min_{u \in F(\mathbb{R}^k, \mathbb{R}^m)} L(u) \]  
(unconstrained minimization problem with functions as domain)

\[ x^* = \arg \min_{x \in M} L(x) \]  
(constrained minimization problem)

\[ \sim M \]

it is very thin!!
Motivation

- Many problems are formulated in the domain of a manifold
- Some in particular refers to the set of the rigid motions $SE(3)$

$$\begin{cases} 
\min f(x, \mathcal{O}) \\
x \in SE(3) 
\end{cases}$$

$\mathcal{O}$ = observations

Manifold (adding noise to a rotation matrix gives as result a non rotation matrix)

Motivation

- Rigid Registration
  - Camera pose estimation
    - **Input:** two images (with known intrinsics)
      - Compute correspondences between these images
      - Estimate the essential matrix $p_i^T E p_i = 0$
      - Factorize E in (R,t)
      - Compute the 3D structure
      - Bundle-Adjustment

\[
\min_{R, t, M^j} \sum_{j=1}^{n} d\left(K \begin{bmatrix} R & t \end{bmatrix} M^j, m^j\right)^2
\]
Motivation

- The trajectory of a rigid object

\[ \gamma : \mathbb{R} \to SE(3) \]

is a (smooth) curve in \( SE(3) \)

- 3D Rigid Object or Camera Tracking

\[ \begin{align*}
\min f(\gamma, O) \\
\gamma : \mathbb{R} \to SE(3)
\end{align*} \]
Motivation

- **Rigid Motion Interpolation**
  - Given two rigid motions: $A$ and $B \in SE(3)$
  - Find a smooth rigid motion $\gamma$ connecting $A$ and $B$ (or find the shortest path between $A$ and $B$)
Content

- Rigid transformations
- Linear Matrix Groups
- Manifolds
- Lie Groups/Lie Algebras
- Charts on SO(2) and SO(3)
Rigid Transformations

\( F : A \rightarrow A \) is a transformation
Rigid Transformations

$F : A \rightarrow A$ is a rigid transformation iff,

- it preserves distances $d(x, y) = d(F(x), F(y)), \forall x, y \in A$ (isometry)
- it preserves the space orientation (no reflection)
Taxonomy

Affine maps

Affine + Conformal

Conformal maps

Isometry

Linear

Rigid

Isometries which do not preserve the orientation
if $A$ is a finite dimensional space (e.g. $\mathbb{R}^n$)
a rigid transformation $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$
can be written as

$$F(x) = Rx + t$$

$x \in \mathbb{R}^n$
$t \in \mathbb{R}^n$

$R \in \mathbb{R}^{n \times n}$

- $R$ orthogonal (isometry)
- $\det(R) = 1$ (preserve orientation)

$F(x) = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} x$

$x \in \mathbb{RP}^n$

Rotation matrix

Projective space

Note: in this space, $F$ is also linear
The set of all the rigid transformations in $\mathbb{R}^n$ is a **group** (not commutative) with the composition operation

$\left( \left\{ F : \mathbb{R}^n \to \mathbb{R}^n \mid F \text{ rigid } \right\}, \circ \right)$

This set is isomorphic to the **special Euclidean group** $SE(n)$

The existence of an isomorphism is important because one can represent each rigid transformation as an element of $SE(n)$ (bijective) and performs operations in this latter space (which will correspond to operations in the former space)
Content

- Rigid transformations
- **Matrix Groups**
- Manifolds
- Lie Groups/Lie Algebras
- Charts on SO(2) and SO(3)
Matrix Groups

- The set of all the nxn invertible matrices is a group w.r.t. the matrix multiplication

\[ GL(n) = \left( \{ M \in \mathbb{R}^{n \times n} \mid \det(M) \neq 0 \} , \times \right) \]

General linear group

- \( GL(n) \) is isomorphic to the group of linear and invertible transformations in \( \mathbb{R}^n \) with the composition as operation

\[ \left( \{ F : \mathbb{R}^n \to \mathbb{R}^n \mid F \text{ linear bijective} \} , \circ \right) \]

- It exists an isomorphism \( \Psi(x \to Mx) = M \), such that

\[ \Psi(F \circ G) = \Psi(F) \times \Psi(G) \]
Matrix Groups

- The set of all the nxn orthogonal matrices is a group w.r.t. the matrix multiplication

\[ O(n) = \left\{ A \in GL(n) \mid A^{-1} = A^T \right\}, \times \]

**Orthogonal group**

- \( O(n) \) is isomorphic to the group of **linear isometries** in \( \mathbb{R}^n \) with the composition as operation

\[ \left\{ F : \mathbb{R}^n \to \mathbb{R}^n \mid F \text{ linear isometry} \right\}, \circ \]

- **PS:** \( A \in O(n) \Rightarrow \det(A) = \pm 1 \)
Matrix Groups

- The set of all the nxn orthogonal matrices with determinant equal to 1 is a group w.r.t. the matrix multiplication

\[ SO(n) = \{ A \in O(n) \mid \det(O) = +1 \}, \times \]  

Special orthogonal group

- \( SO(n) \) is isomorphic to the group of **linear rigid transformations** in \( \mathbb{R}^n \) with the composition as operation

- It exists an isomorphism \( \Psi(x \rightarrow Mx) = M \), such that

\[ \Psi(F \circ G) = \Psi(F') \times \Psi(G') \]
Groups of Matrices: Summary

\[ GL(n) = \{ M \in \mathbb{R}^{n \times n} \mid det(M) \neq 0 \}, \times \]  General linear group of order n

\[ O(n) = \{ A \in GL(n) \mid A^{-1} = A^T \}, \times \]  Orthogonal group of order n

\[ SO(n) = \{ A \in O(n) \mid det(O) = +1 \}, \times \]  Special orthogonal group of order n

\[ O(n)/SO(n) = \{ A \in O(n) \mid det(O) = -1 \} \]  Set of orthogonal matrices which do not preserve orientation (not a group)

\[ \mathbb{R}^{n \times n} = \text{vector space of all the nxn matrices} \]
SO(n) in practice

\[ \begin{align*}
M &\in SO(3) \\
M &= \begin{bmatrix}
\cdot & \cdot & \cdot \\
\cdot & v_2 & v_3 \\
\cdot & \cdot & \cdot
\end{bmatrix}
\end{align*} \]

Orthogonality:
\[ <v_i, v_j> = 0 \]
\[ |v_i| = 1 \]

Canonical basis

Coordinates of the rotated \( p \) in the canonical basis

\[ F(p) = Mp \]
Special Euclidean group

- The Cartesian product $SO(n) \times \mathbb{R}^n$ is a group w.r.t. a “weird” operation

$$SE(n) = (SO(n) \times \mathbb{R}^n, \times)$$

- The “weird” operation is define in such a way that the group $SE(n)$ is isomorphic to the group of \textit{rigid transformations} in $\mathbb{R}^n$ with the composition as operation

- It exists an isomorphism $\Psi(x \to Rx + t) = (R, t)$, such that

$$\Psi(F \circ G) = \Psi(F) \times \Psi(G)$$

$$F(x) = Mx + t$$
$$G(x) = Sx + q$$
$$\Psi(F \circ G) = (M, t) \times (S, q)$$

Commutative??
The Geometry of these Groups

- GL(n), O(n), SO(n) and SE(n) are all subset of a vector space

- GL(n), O(n), SO(n) and SE(n) are all smooth manifolds
  (surfaces, curves, solids, etc... immerse in some big vector space)
SO(2) and SO(3): Shape

- What are the shapes of these two manifolds?
GL(N), O(N) and SO(N)

O(n) is the union of these two manifolds
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The concept of manifold generalizes
- the concepts of **curve**, **area**, **surface**, and **volume** in the Euclidean space/plane
- ... but not only ...

A manifold does not have to be a subset of a bigger space, it is an object on its own.

A manifold is one of the most generic objects in math..

Almost everything is a manifold
**Differential Manifold**

- **Manifold** = topological set + a set of charts

\[ M = (S, T, A) \]

**Atlas** = set of charts

A chart on a manifold is bijective, continuous, and with a continuous inverse. It is sufficiently regular: bijective, derivable \( s \) times with inverse derivable \( s \) times.

**Chart**:
- Bijective, continuous, and with continuous inverse.
The **tangent space of** $M$ **in** $p$ is the set of all the smooth curves in $M$ of type

$$\gamma : \mathbb{R} \rightarrow M$$

where

- $\gamma \in C^0$
- $\gamma(0) = p$

- grouped accordingly to their first derivative in $p$
The tangent space of $M$ in $p$ is isomorphic to a subspace of $V$.

It corresponds to the velocity of $\gamma$ in $p$ (direction and speed).
The tangent space of $M$ in $p$ is isomorphic to a subspace of $V$.

- $TM(p)$ is a vector space (subspace of $V$)
  - has dimension $k$

1-manifold $\rightarrow$ 1 dim TM (curves) (lines)
2-manifold $\rightarrow$ 2 dim TM (surfaces) (planes)
3-manifold $\rightarrow$ 3 dim TM (volumes) (full volumes)
What are the tangent spaces of these two manifolds?

- $\mathbb{SO}(2)$: vector space with 1 dimension, subspace of $\mathbb{R}^{2 \times 2}$
- $\mathbb{SO}(3)$: vector space with 3 dimensions, subspace of $\mathbb{R}^{3 \times 3}$

They are matrices.
Skew-Symmetric Matrix

M is skew-symmetric matrix iff $M^T = -M$

$$
\begin{bmatrix}
0 & 3 & 6 \\
-3 & 0 & -1 \\
-6 & 1 & 0
\end{bmatrix}
\quad
\begin{bmatrix}
0 & 4 \\
-4 & 0
\end{bmatrix}
$$

$so(n) = \langle \{ M \in \mathbb{R}^{n \times n} \mid M^T = -M \} , +, \cdot, [\ ] \rangle$

Special orthogonal Lie algebra (vector space with Lie brackets)

$$[A, B] = AB - BA$$
The Special orthogonal Lie algebra is the tangent space of $SO(n)$ at the identity.

$SO(n)$ is a vector space so it passes through the null matrix.

so in reality
The Special orthogonal Lie algebra is the tangent space of $SO(n)$ at the identity.

$T_{SO(n)}(I) = so(n)$

valid only at the identity

The tangent space of $SO(n)$ in any other point $R$ is a rotated version of $so(n)$

$T_{SO(n)}(R) = R \times so(n)$

they are no more skew-symmetric matrices but rotations of them
so(2) and so(3)

- $so(3)$ is a vector space of dimension 3
- $so(2)$ is a vector space of dimension 1

An element in $so(3)$ or $so(2)$ represents an infinitesimal rotation from the identity matrix.
The hat operator

- The hat operator in $so(3)$

\[ \hat{\cdot} : \mathbb{R}^3 \rightarrow so(3) \]

\[ (x, y, z) \rightarrow \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \]

- It is an isomorphism from $so(3)$ to $\mathbb{R}^3$ (it maps + into +)

- The hat operator in $so(2)$

\[ \hat{\cdot} : \mathbb{R} \rightarrow so(2) \]

\[ \hat{x} \rightarrow \begin{bmatrix} 0 & -x \\ x & 0 \end{bmatrix} \]
The hat operator

- The hat operator is used to define cross-product in matrix form:

\[ a \times b = \hat{a}\hat{b} \quad \forall a, b \in \mathbb{R}^3 \]

- The hat operator maps cross products into [..,]
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Lie Groups

- GL(n), O(n), SO(n) and SE(n) are all **Lie groups**
  (groups which are also smooth manifold where the operation is a differentiable function between manifolds)
Exponential Map

- Given a Lie group $G$, with its related Lie Algebra $g = T_G(I)$, there always exists a smooth map from Lie Algebra $g$ to the Lie group $G$ called **exponential map**

  $$\exp : g \rightarrow G$$

- Diagram showing the relationship between $\text{SO}(n)$, $\text{so}(n)$, and the exponential map $\exp : \text{so}(n) \rightarrow \text{SO}(n)$.
Exponential Map

- Given a Lie group $G$, with its related Lie Algebra $g = TG(I)$, there always exists a smooth map from Lie Algebra $g$ to the Lie group $G$ called **exponential map**

$$\exp : g \rightarrow G$$

$\exp(A) = \text{is the point in } G \text{ that can be reached by traveling along the geodesic passing through the identity } I \text{ in direction } A$, for a unit of time

(Note: $A$ defines also the traveling speed)
Exponential Map

- The exponential map for any matrix Lie group (GL(n), O(n), and SO(n)) coincides with the matrix exponential:

\[ \exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k \]

- It is a **smooth map**
- It is **surjective** (it covers the Lie Group entirely)
- It is **not injective** (is a many to one map)
Exponential Map and Hat Operator

- composed with the hat operator, it is a smooth and surjective map from $\mathbb{R}^k$ to $SO(n)$ (k = the dimension of the tangent space)

$$\omega \in \mathbb{R}^k, \quad \hat{\omega} \in so(n), \quad \exp(\hat{\omega}) = \sum_{k=0}^{\infty} \frac{1}{k!} \hat{\omega}^k \in SO(n)$$

Angle-Axis representation

$$\omega \in \mathbb{R}^k \rightarrow \exp(\hat{\omega})$$
Properties

\[ \hat{e}^0 = e^0 = I \]
\[ e^{-X} = (e^X)^{-1} \]
\[ e^{X+Y} \neq e^X e^Y \]

Identity

Inverse \[ \rightarrow \quad e^{-\hat{\omega}} = e^{-\hat{\omega}} = (e^{\hat{\omega}})^{-1} \]

in general not “Linear” \hspace{1cm} (different from the standard exp in \( \mathbb{R} \))

\[ e^{sX + tX} = e^{sX} e^{tX} \]

\( \forall t, s \in \mathbb{R} \)

Derivative

\[ \partial e^X = \partial X e^X = e^X \partial X \]
Physical meaning of $so(3)$

\[ p(t) = e^{\bar{\omega}(t)} p(0) \]

Position

\[ \frac{\partial p}{\partial t}(t) = \frac{\partial \bar{\omega}}{\partial t}(t) e^{\bar{\omega}(t)} p(0) = \frac{\partial \bar{\omega}}{\partial t}(t) p(t) \]

Velocity

Spatial (angular) velocity $\in so(3)$

(transform each point in $\mathbb{R}^3$ into the corresponding speed that that point undergoes during the rotation at time $t$)
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Charts on $\text{SO}(2)$ and $\text{SO}(3)$
Charts on $SO(2)$

- Chart of $SO(2)$, that cover the entire $SO(2)$ using a single parameter

\[
\gamma : [0, 2\pi) \to SO(2)
\]

\[
\gamma(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \in SO(2)
\]

\[
\min f(T, \mathcal{O}) \quad T \in SO(2) \quad \iff \quad \min f(\gamma(\theta), \mathcal{O}) \quad \theta \in [0, 2\pi)
\]
**Euler’s Theorem for rotations**: Any element in SO(3) can be described as a sequence of three rotations around the canonical axes, where no successive rotations are about the same axis.

\[
R_x(\alpha) = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos(\alpha) & -\sin(\alpha) \\
0 & \sin(\alpha) & \cos(\alpha)
\end{bmatrix}
\]

\[
R_y(\beta) = \begin{bmatrix}
\cos(\beta) & 0 & \sin(\beta) \\
0 & 1 & 0 \\
-\sin(\beta) & 0 & \cos(\beta)
\end{bmatrix}
\]

\[
R_z(\gamma) = \begin{bmatrix}
\cos(\gamma) & -\sin(\gamma) & 0 \\
\sin(\gamma) & \cos(\gamma) & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

For any \( R \in SO(3) \) there \( \exists \alpha, \beta, \gamma \mid R = R_x(\alpha)R_y(\beta)R_z(\gamma) \)

\( \alpha, \beta, \gamma \) are called **Euler angles** of \( R \) according to the XYZ representation.
SO(3): Euler angles

- Given $M$ there are 12 possible ways to represent it

$$M \in SO(3) \iff \exists \alpha, \beta, \gamma \mid M = R_x(\alpha)R_y(\beta)R_z(\gamma)$$

- Unfortunately, all of them have the same drawbacks!! (see later)

- A common representation is ZYX corresponding to a rotation first around the x-axis (roll), then the y-axis (pitch) and finally around the z-axis (yaw).

Remarks: multiplication is not commutative
The parameterization is non-linear

The parameterization is modular

\[ R_x(\alpha + 2k\pi) = R_x(\alpha) \]

(but this is something that we need to live with in any representation of SO(3))

Beside the modularity, the parameterization is not unique:

for some \( R \) in SO(3), \( \exists \alpha_1, \beta_1, \gamma_1 \) and \( \alpha_2, \beta_2, \gamma_2 \) such that

\[
M = R_x(\alpha_1)R_y(\beta_1)R_z(\gamma_1)
\]

\[
M = R_x(\alpha_2)R_y(\beta_2)R_z(\gamma_3)
\]
SO(3): Euler angles

- The parameterization have some singularities, called **gimbal lock**

- A gimbal lock happens when after a rotation around an axis, two axes align, resulting in a loss of one degree of freedom

\[ (a) \quad \begin{array}{c}
\includegraphics[width=0.3\textwidth]{gimbal_lock_a}
\end{array} \]

\[ (b) \quad \begin{array}{c}
\includegraphics[width=0.3\textwidth]{gimbal_lock_b}
\end{array} \]

\textit{cannot move (lock)}
SO(3): Euler angles

- The parameterization have some singularities, called **gimbal lock**

- A gimbal lock happens when after a rotation around an axis, two axes align, resulting in a loss of one degree of freedom

- The name gimbal lock derives from the gimbal

- Even the most advanced modeling software uses Euler angles to parameterize the orientation of the rendering window. This is because Euler angles are more intuitive to the user. As a drawback, the gimbal lock is often noticeable.
Euler Angles and Angle/Axis

- The Euler angle representation say

\[ R \in SO(3) \iff \exists \alpha, \beta, \gamma \mid R = R_x(\alpha)R_y(\beta)R_z(\gamma) \]

\[ = e^{\alpha \hat{x}} e^{\beta \hat{y}} e^{\gamma \hat{z}} \]

\[ \neq e^{\alpha \hat{x} + \beta \hat{y} + \gamma \hat{z}} \]

- while Euler angle define 3 rotation matrices, the angle/axis representation define a single rotation matrix identified by an element of \( \mathbb{R}^3 \)