Rigid Transformations

--- the geometry of SO(3) & SE(3) ---

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Motivation

$$x^* = \arg \min_{x \in \mathbb{R}^k} L(x)$$

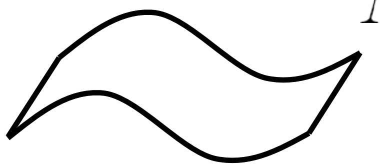
(unconstrained minimization problem)

$$u^* = \arg\min_{u \in \mathbb{F}(\mathbb{R}^k, \mathbb{R}^m)} L(u)$$

(unconstrained minimization problem with functions as domain)

$$x^* = \arg \min_{x \in M} L(x)$$

(constrained minimization problem)



M

it is very thin!!

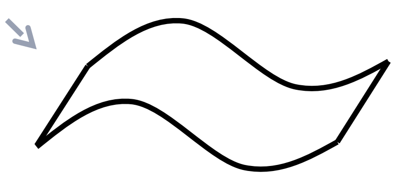
Motivation

- Many problems are formulated in the domain of a manifold
- Some in particular refers to **the set of the rigid motions** $SE(3)$

$$\begin{cases} \min f(x, \mathcal{O}) \\ \quad x \in SE(3) \end{cases}$$

$\mathcal{O}$ = observations

optimization in the domain of a manifold



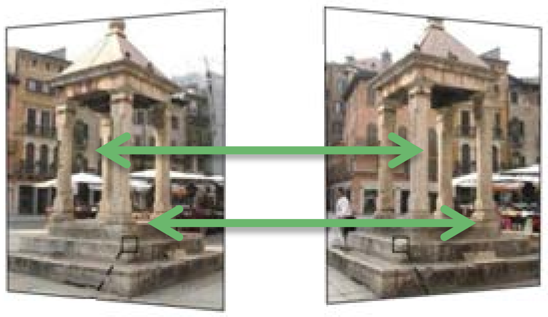
Manifold
(adding noise to a rotation matrix gives as result a non rotation matrix)

- **Reference book:** R. Murray, Z. Li and S. Sastry, “A Mathematical Introduction to Robotic Manipulation”, CRC Press 1994

Motivation

- Rigid Registration

- Camera pose estimation
  - **Input:** two images (with known intrinsics)
    - Compute correspondences between these images
    - Estimate the essential matrix $\mathbf{p}_i'^{\top} E \mathbf{p}_i = 0$
    - Factorize E in (R,t)
    - Compute the 3D structure
    - Bundle-Adjustment



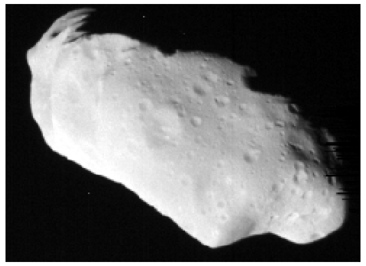
$$\min_{R,\mathbf{t},\mathbf{M}^j} \sum_{j=1}^{n} d(K[R|\mathbf{t}]\mathbf{M}^j, \mathbf{m}^j)^2$$

Motivation

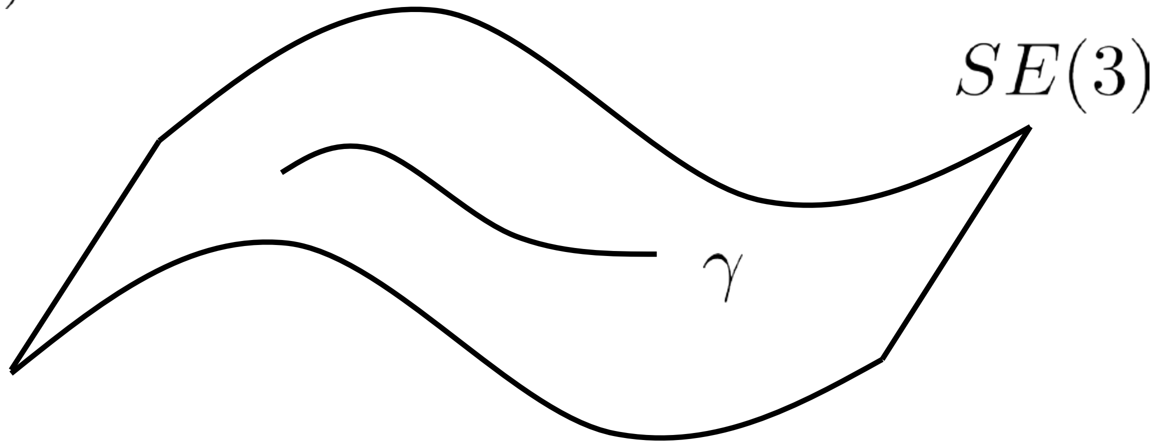
- The trajectory of a rigid object

$$\gamma : \mathbb{R} \to SE(3) \qquad \text{is a (smooth) curve in } SE(3)$$

- 3D Rigid Object or Camera Tracking

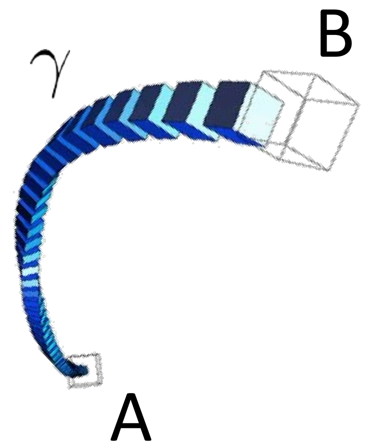


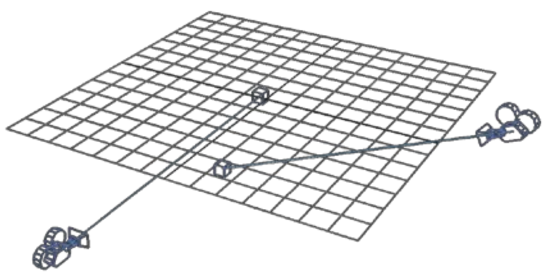
$$\begin{cases} \min f(\gamma, \mathcal{O}) \\ \qquad \gamma : \mathbb{R} \to SE(3) \end{cases}$$

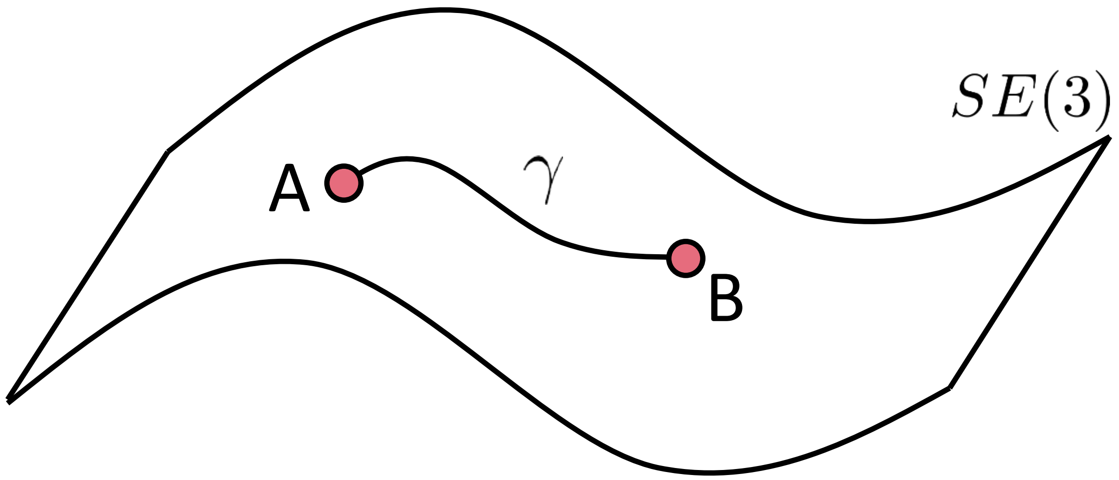


Motivation

- Rigid Motion Interpolation
  - Given two rigid motions: A and B $\in SE(3)$







  - Find a smooth rigid motion γ connecting A and B (or find the shortest path between A and B)

Content

- **Rigid transformations**
- Linear Matrix Groups
- Manifolds
- Lie Groups/Lie Algebras
- Charts on SO(2) and SO(3)

Rigid Transformations

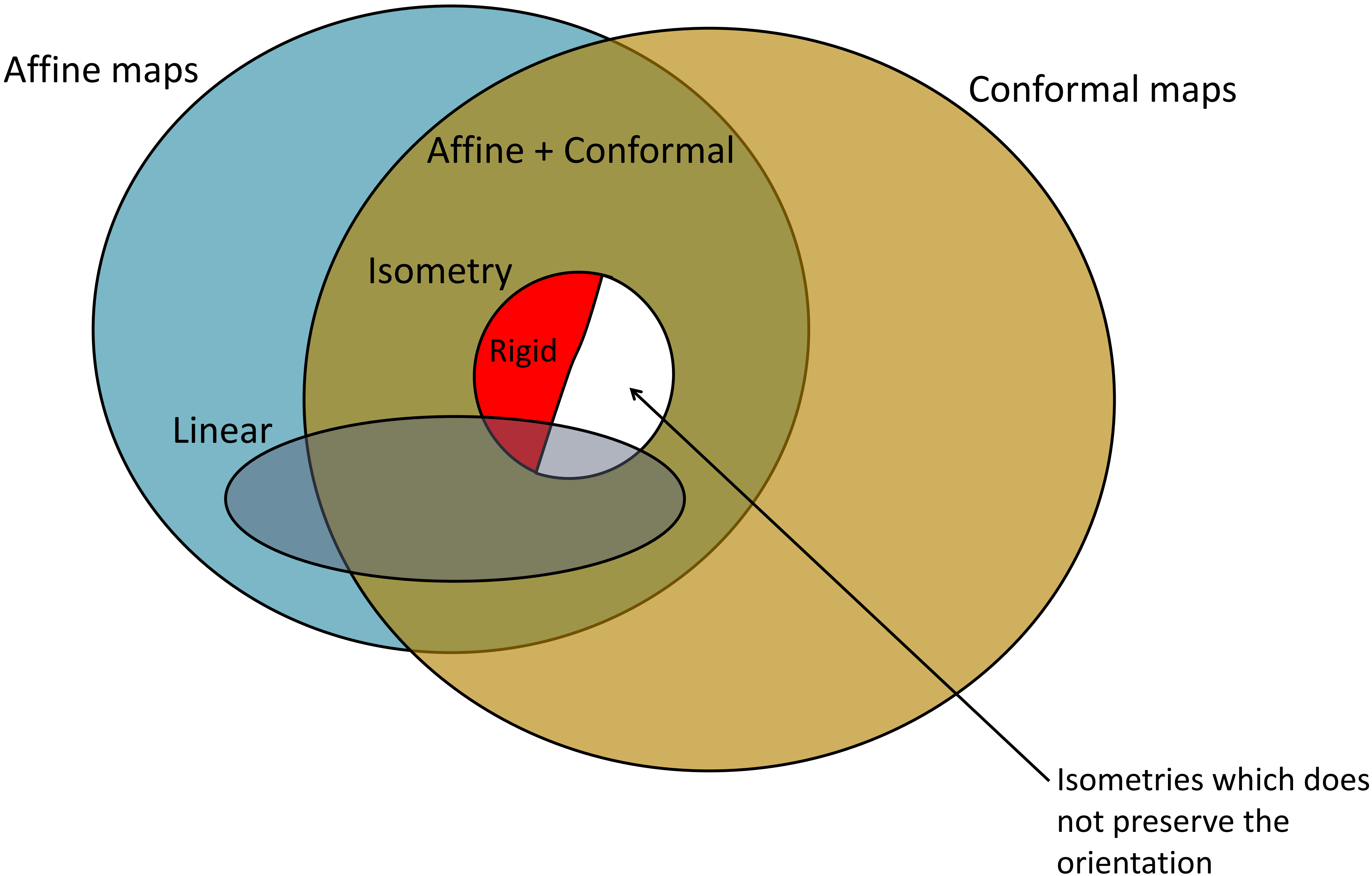
$F : A \to A$ is a transformation

Rigid Transformations

$F : A \to A$ is a rigid transformation iff,

- it preserves distances $d(x,y) = d(F(x), F(y)), \quad \forall x, y \in A$ (isometry)
- it preserves the space orientation (no reflection)

Taxonomy


Affine maps
Conformal maps
Affine + Conformal
Isometry
Rigid
Linear
Isometries which does not preserve the orientation

Representation

if A is a finite dimensional space (e.g. $\mathbb{R}^n$)

a rigid transformation $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$

can be written as

$$F(x) = Rx + t \qquad x \in \mathbb{R}^n$$

$$t \in \mathbb{R}^n$$

$$R \in \mathbb{R}^{n \times n}$$

- R orthogonal (isometry)
- $det(R) = 1$ (preserve orientation) $\boxed{*}$

Rotation matrix

$$F(x) = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} x \qquad x \in \mathbb{RP}^n$$

Projective space

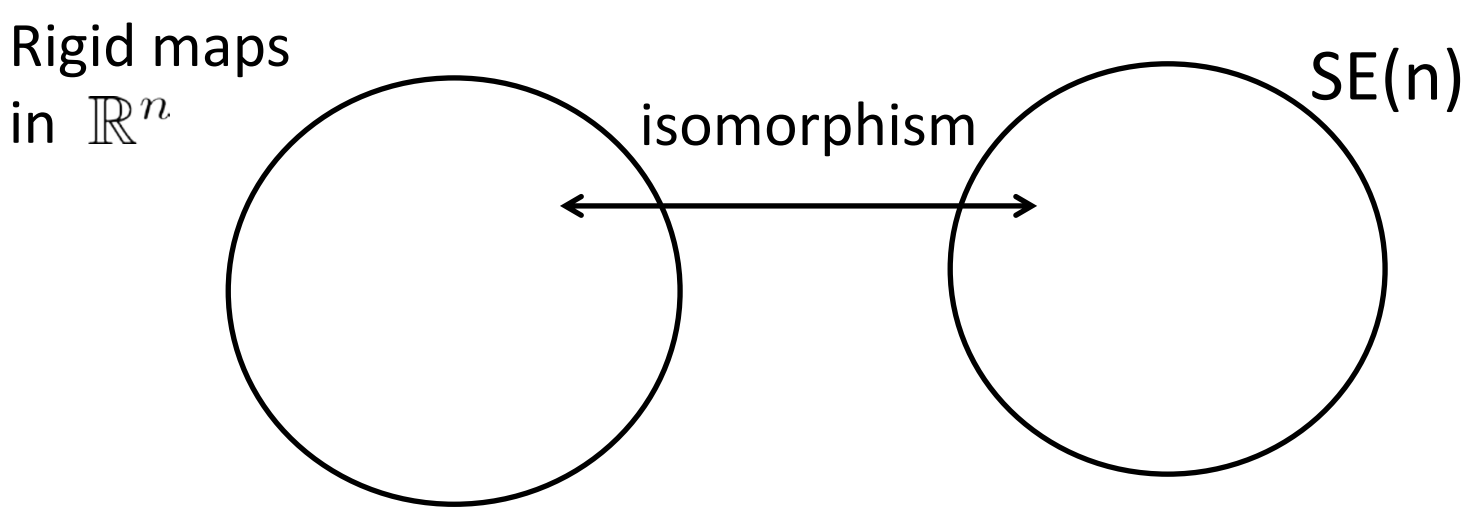
Note: in this space, F is also linear

Rigid Transformations

- The set of all the rigid transformations in $\mathbb{R}^n$ is a **group** (not commutative) with the composition operation

$$(\{F : \mathbb{R}^n \to \mathbb{R}^n \mid F \text{ rigid } \}, \circ)$$

- This set is isomorphic to the **special Euclidean group SE(n)**



- The existence of an isomorphism is important because one can represent each rigid transformation as an element of SE(n) (bijective) and performs operations in this latter space (which will correspond to operations in the former space)

Content

- Rigid transformations
- **Matrix Groups**
- Manifolds
- Lie Groups/Lie Algebras
- Charts on SO(2) and SO(3)

Matrix Groups

- The set of all the nxn invertible matrices is a group w.r.t. the matrix multiplication

$$GL\left(n\right) = \left(\left\{M \in \mathbb{R}^{nxn} \mid det(M) \neq 0\right\}, \times\right)$$ **General linear group**

- GL(n) is isomorphic to the group of **linear and invertible transformations** in $\mathbb{R}^n$ with the composition as operation

$$(\{F : \mathbb{R}^n \rightarrow \mathbb{R}^n \mid F \text{ linear bijective}\}, \circ)$$

- It exists an isomorphism $\Psi(x \rightarrow Mx) = M$, such that

$$\Psi(F \circ G) = \Psi(F) \times \Psi(G)$$

*

Matrix Groups

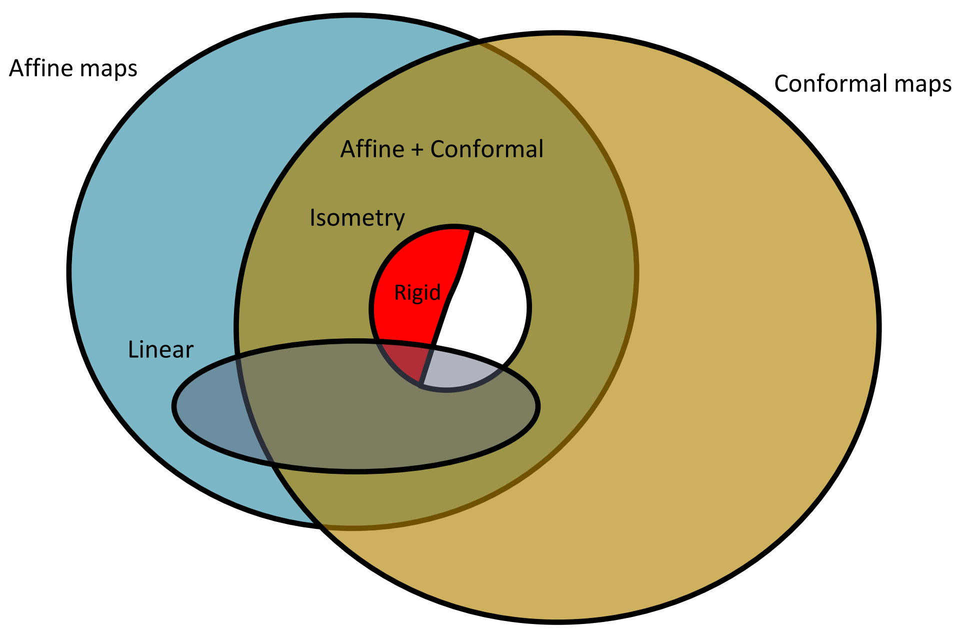
- The set of all the nxn orthogonal matrices is a group w.r.t. the matrix multiplication

$$O\left(n\right) = \left(\left\{A \in GL\left(n\right) \mid A^{-1} = A^{T}\right\}, \times\right)$$ **Orthogonal group**

- O(n) is isomorphic to the group of **linear isometries** in $\mathbb{R}^n$ with the composition as operation

$$(\{F : \mathbb{R}^n \to \mathbb{R}^n \mid F \text{ linear isometry}\}, \circ)$$

- PS: $A \in O(n) \Rightarrow \det(A) = \pm 1$



Matrix Groups

- The set of all the nxn orthogonal matrices with determinant equal to 1 is a group w.r.t. the matrix multiplication

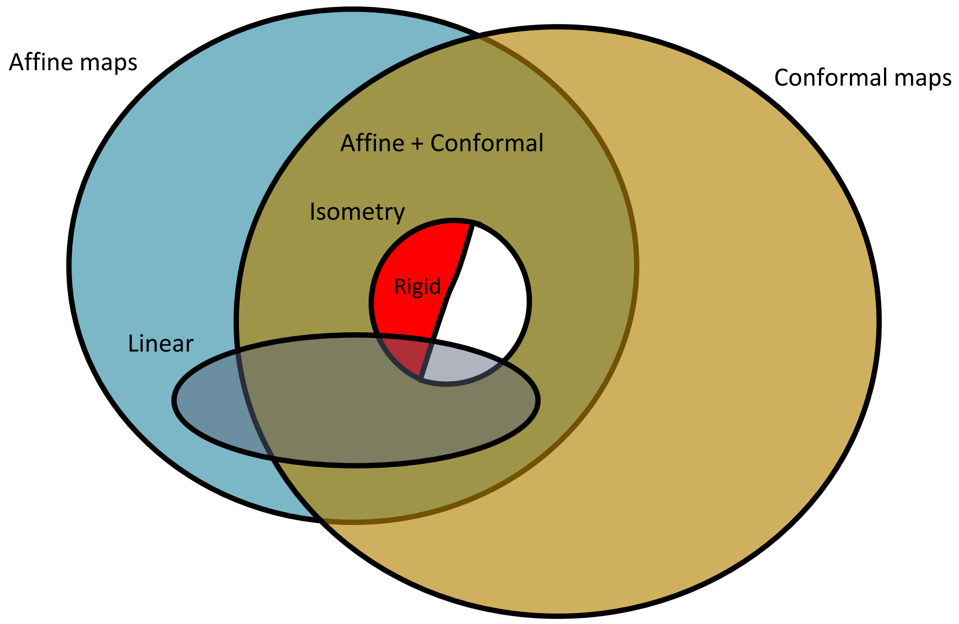
$$SO(n) = (\{A \in O(n) \mid det(O) = +1\}, \times)$$

Special orthogonal group

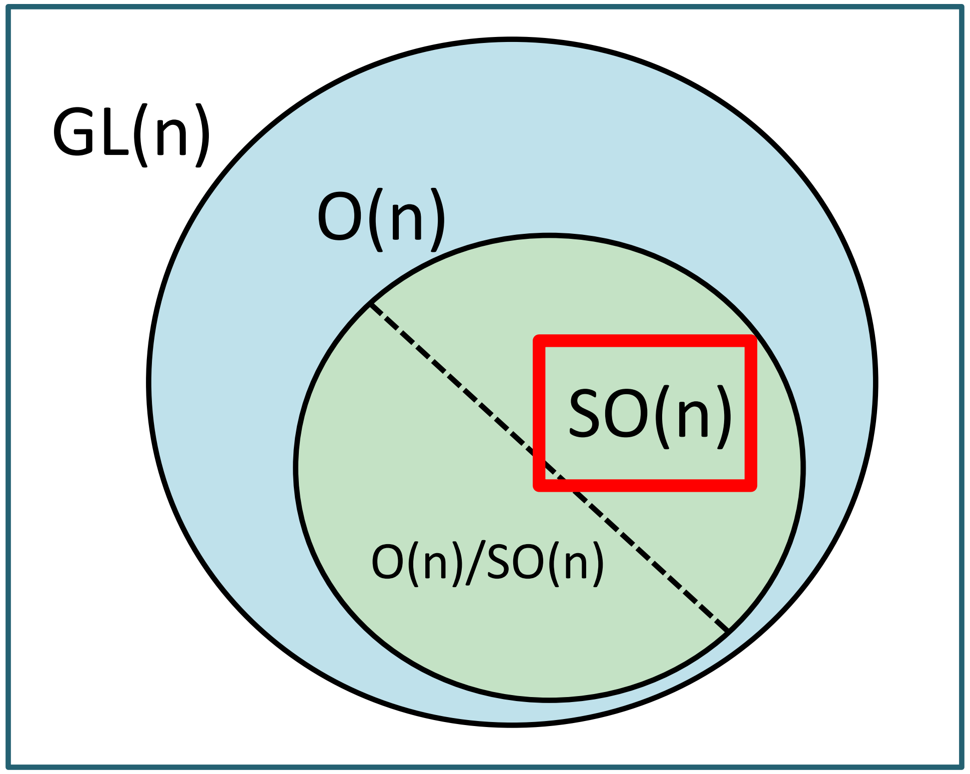
- SO(n) is isomorphic to the group of **linear rigid transformations** in $\mathbb{R}^n$ with the composition as operation

- It exists an isomorphism $\Psi(x \to Mx) = M$, such that

$$\Psi(F \circ G) = \Psi(F) \times \Psi(G)$$



Groups of Matrices: Summary



$\mathbb{R}^{nxn}$ = vector space of all the nxn matrices

$GL\left(n\right)=\left(\left\{M\in\mathbb{R}^{nxn}\mid det(M)\neq0\right\},\times\right)$ General linear group of order n

$O\left(n\right)=\left(\left\{A\in GL\left(n\right)\mid A^{-1}=A^{T}\right\},\times\right)$ Orthogonal group of order n

$\boxed{SO\left(n\right)=\left(\left\{A\in O\left(n\right)\mid det(O)=+1\right\},\times\right)}$ Special orthogonal group of order n

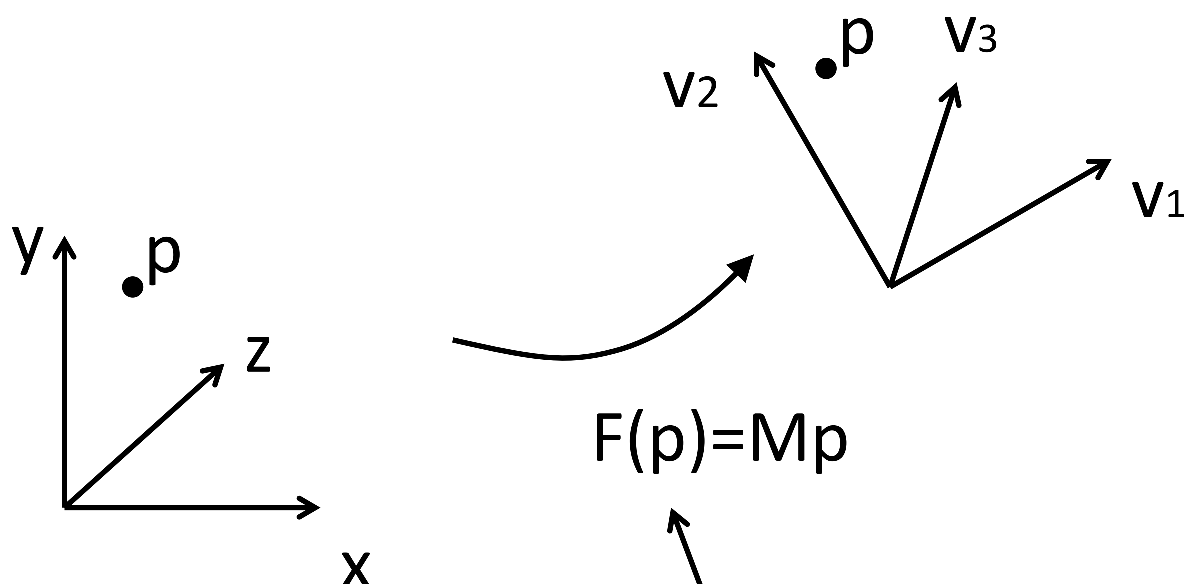
$O(n)/SO(n)=\left\{A\in O\left(n\right)\mid det(O)=-1\right\}$ Set of orthogonal matrices which do not preserve orientation (not a group)

SO(n) in practice

$$M \in SO(3) \quad \Rightarrow \quad M = \begin{bmatrix} . & . & . \\ v_1 & v_2 & v_3 \\ . & . & . \end{bmatrix}$$

Orthogonality:

$$< v_i, v_j >= 0$$
$$|v_i| = 1$$



Canonical basis

$F(p)=Mp$

Coordinates of the rotated p in the canonical basis

Special Euclidean group

- The Cartesian product $SO(n) \times \mathbb{R}^n$ is a group w.r.t. a “weird” operation

$$SE(n) = (SO(n) \times \mathbb{R}^n, \times)$$

Special Euclidean group

$$(M, t) \times (S, q) = (MS, Mq + t)$$

- The “weird” operation is define in such a way that the group SE(n) is isomorphic to the group of **rigid transformations** in $\mathbb{R}^n$ with the composition as operation

- It exists an isomorphism $\Psi(x \to Rx + t) = (R, t)$, such that

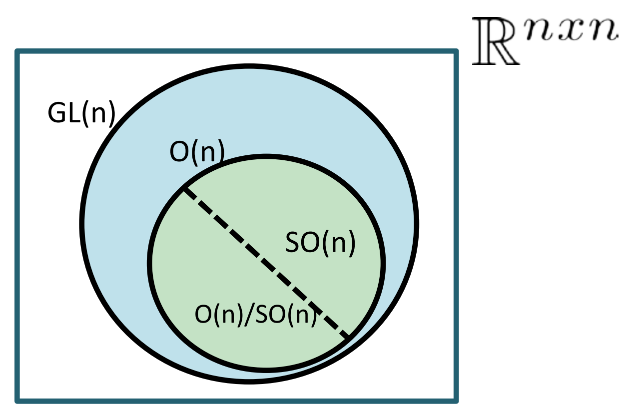
$$\Psi(F \circ G) = \Psi(F) \times \Psi(G)$$

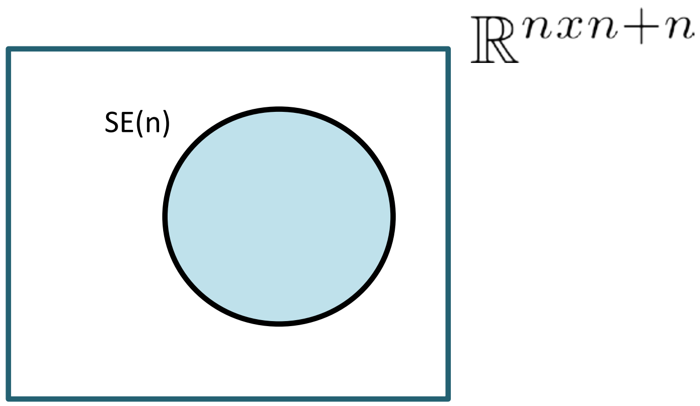
$$\boxed{\begin{array}{l} F(x) = Mx + t \\ G(x) = Sx + q \end{array} \Rightarrow \Psi(F \circ G) = (M, t) \times (S, q)}$$

Commutative??

The Geometry of these Groups

- GL(n), O(n), SO(n) and SE(n) are all subset of a vector space



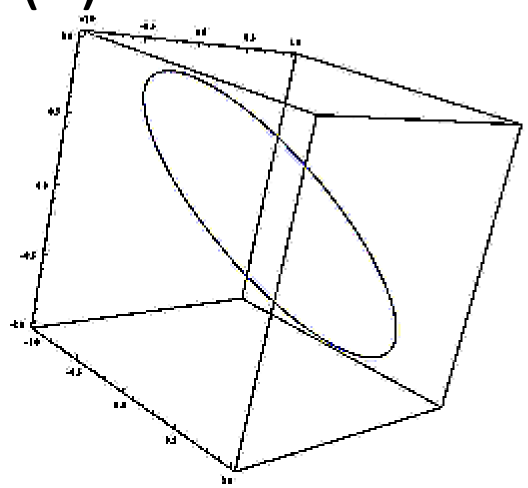


- GL(n), O(n), SO(n) and SE(n) are all **smooth manifolds**
  (surfaces, curves, solids, etc... immerse in some big vector space)

SO(2) and SO(3): Shape

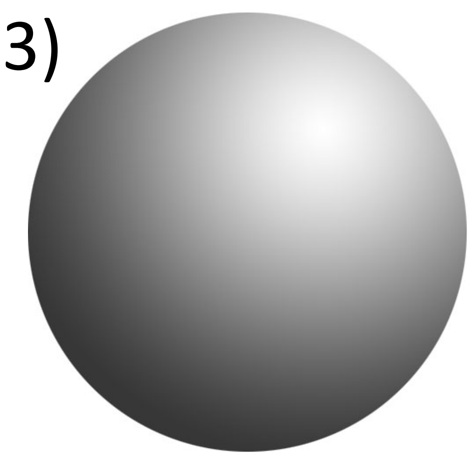
- What are the shapes of these two manifolds?

SO(2)



1-manifold

SO(3)



3-manifold

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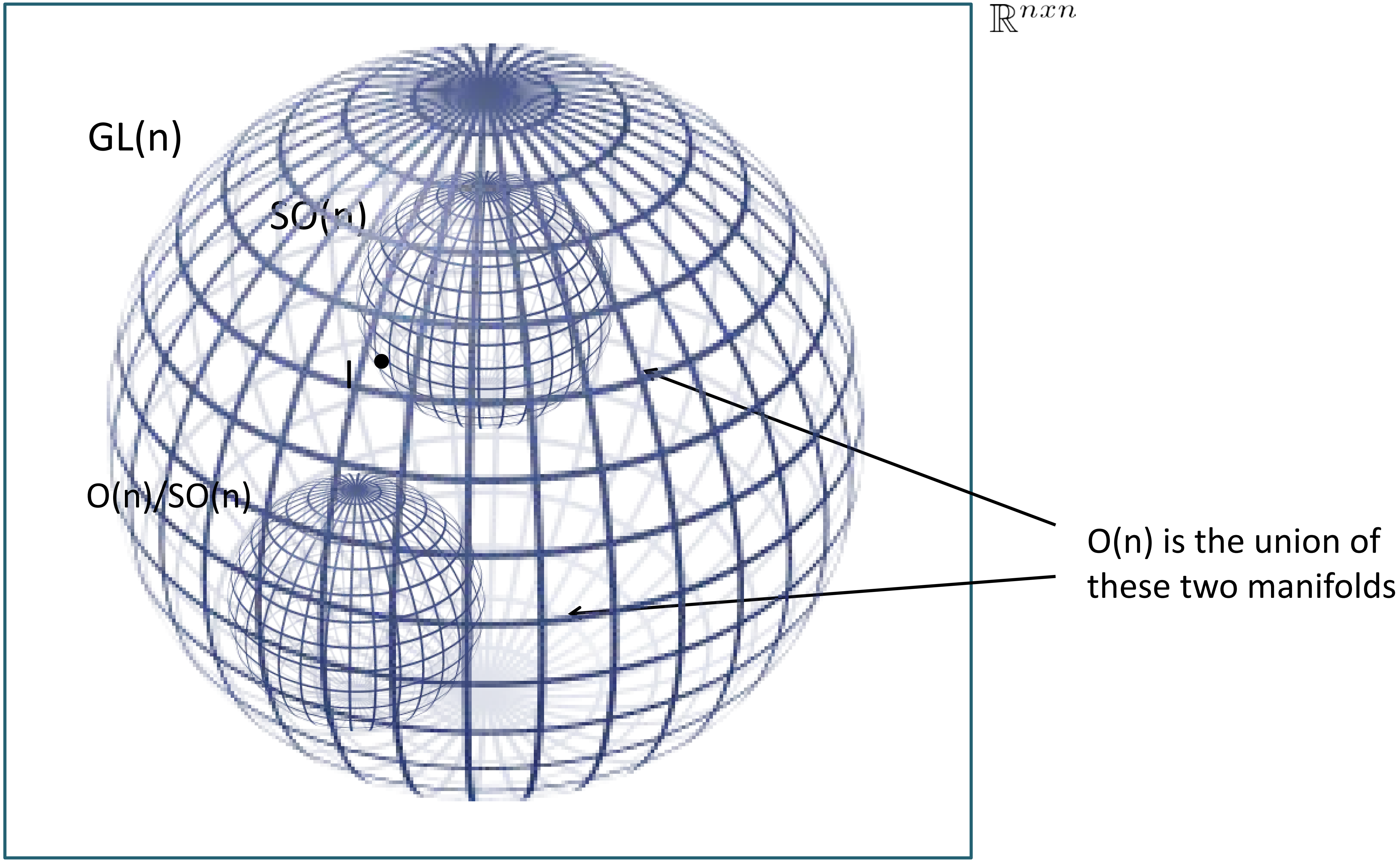
GL(N), O(N) and SO(N)



Content

- Rigid transformations
- Matrix Groups
- **Manifolds**
- Lie Groups/Lie Algebras
- Charts on SO(2) and SO(3)

Manifold

- The concept of manifold generalizes
  - the concepts of **curve**, **area**, **surface**, and **volume** in the Euclidean space/plane
  - … but not only …

- A manifold does not have to be a subset of a bigger space, it is an object on its own.

- A manifold is one of the most generic objects in math..

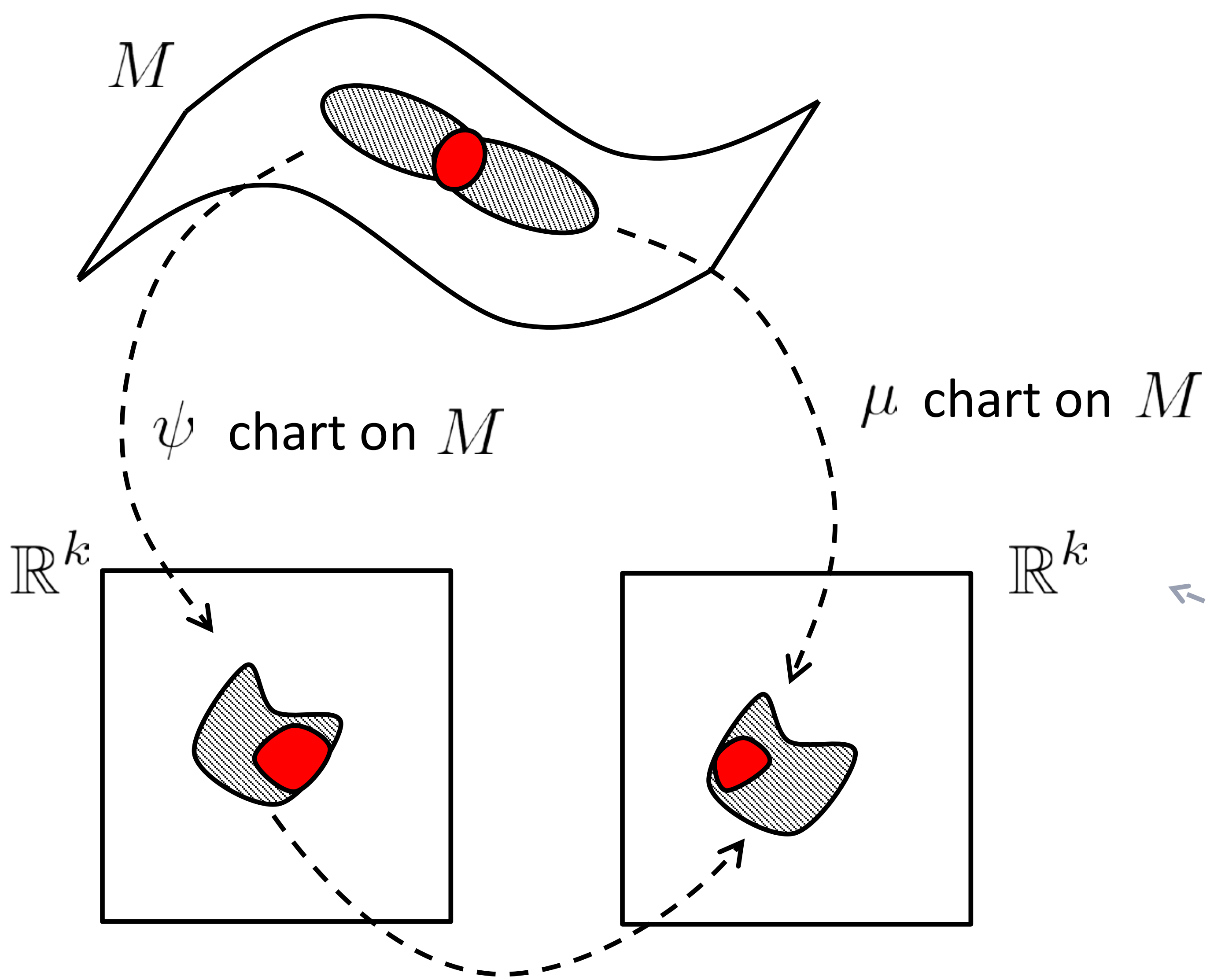
- Almost everything is a manifold

Differential Manifold

- **Manifold =** topological set + a set of charts

$$M = (\underbrace{S, \mathcal{T}}_{\text{topological set}}, \mathcal{A})$$

Atlas = set of charts



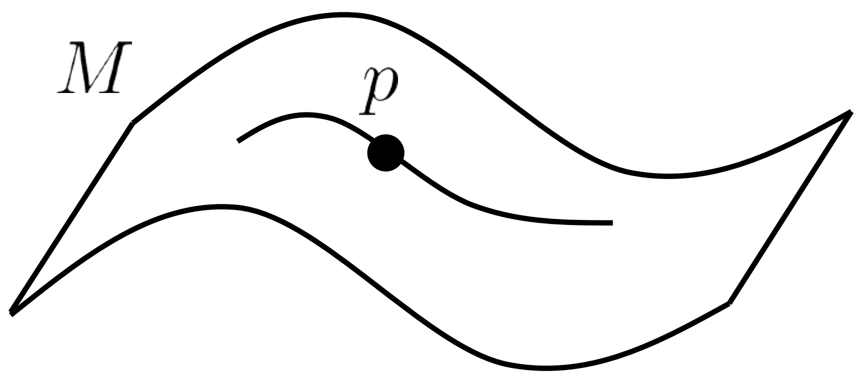
$\mu \circ \psi^{-1}$

sufficiently regular:
Bijective, derivable s times
with inverse derivable s times

M is a **k**-manifold

Chart: bijective, continuous, and with continuous inverse

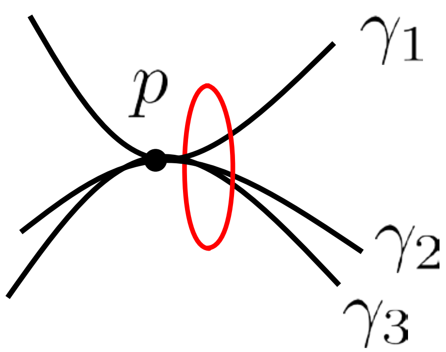
Tangent Space



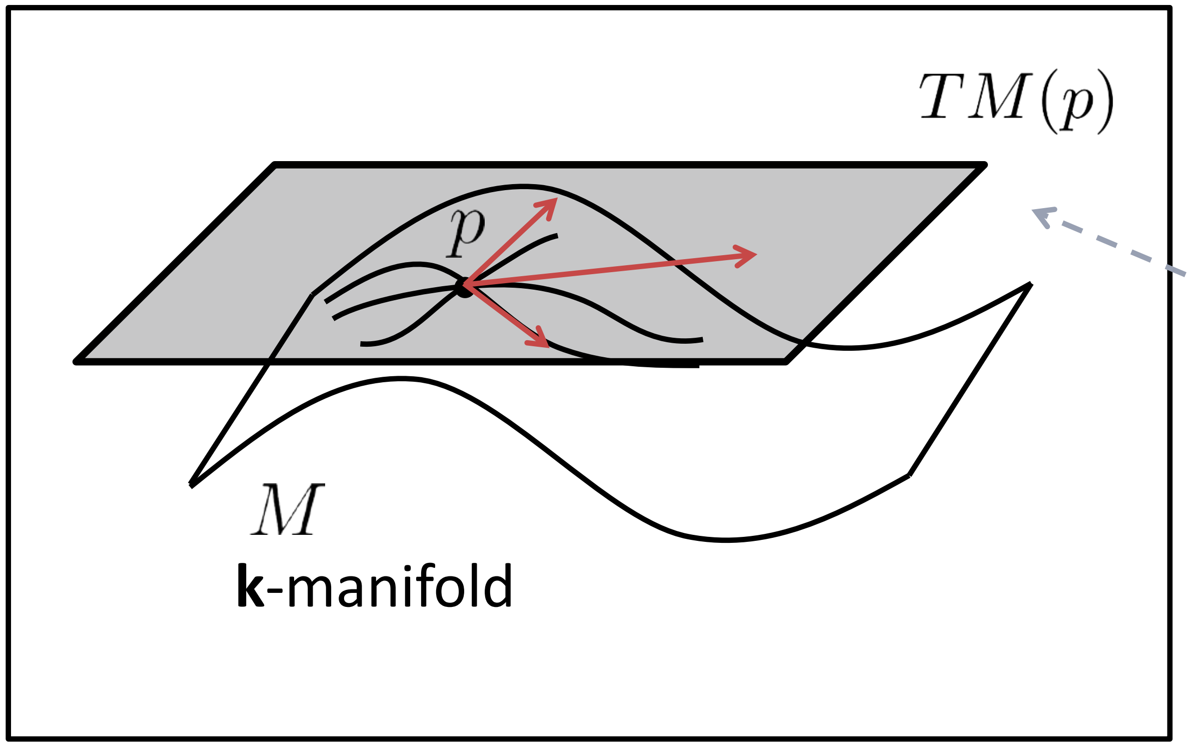
- The **tangent space of** M **in** p is the set of all the smooth curves in M of type

$$\begin{cases} \gamma : \mathbb{R} \rightarrow M \\ \gamma \in C^0 \\ \gamma(0) = p \end{cases}$$

- grouped accordingly to their first derivative in p

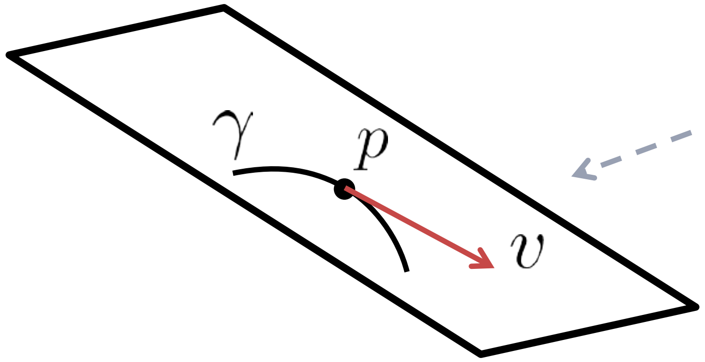


Tangent Space



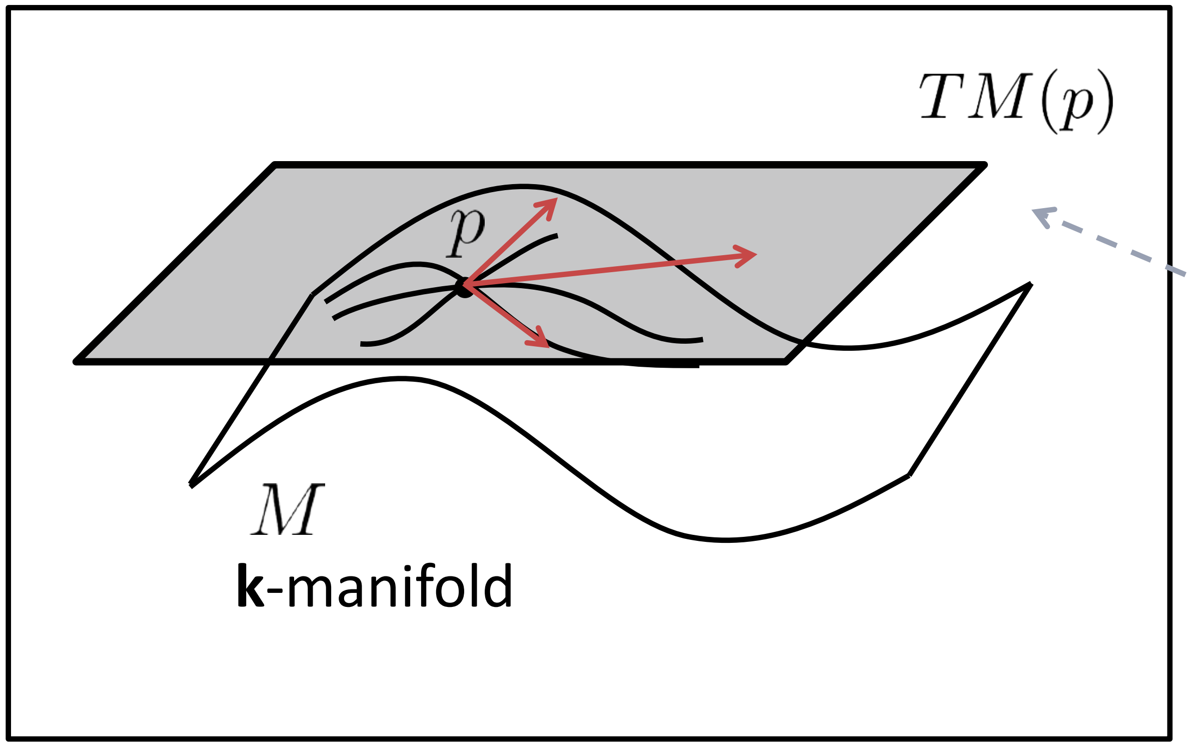
V = Vector space

The tangent space of M in p is isomorphic to a subspace of V



It corresponds to the velocity of γ in p
(direction and speed)

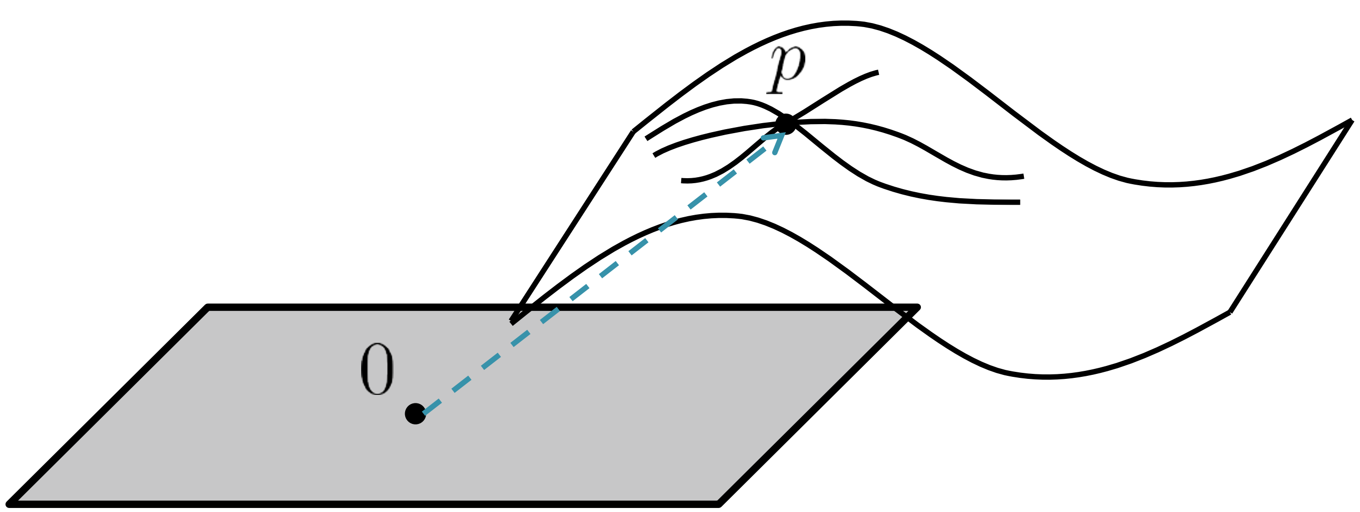
Tangent Space



V = Vector space

The tangent space of M in p is isomorphic to a subspace of V

- $TM(p)$ is a vector space (subspace of V)
  has dimension k



1-manifold (curves) $\longrightarrow$ 1 dim TM (lines)

2-manifold (surfaces) $\longrightarrow$ 2 dim TM (planes)

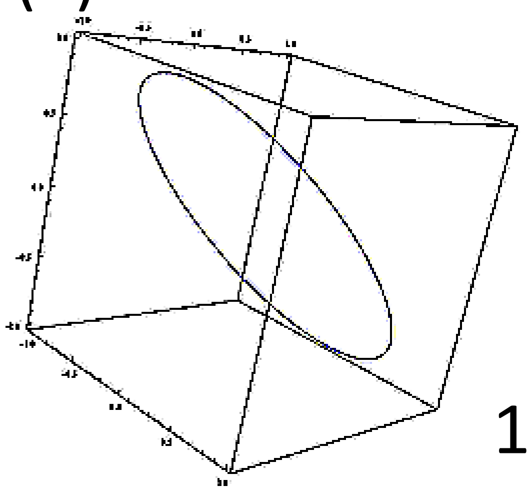
3-manifold (volumes) $\longrightarrow$ 3 dim TM (full volumes)

SO(2) and SO(3): Tangent Spaces

- What are the tangent spaces of these two manifolds?

$\mathbb{R}^{2x2}$

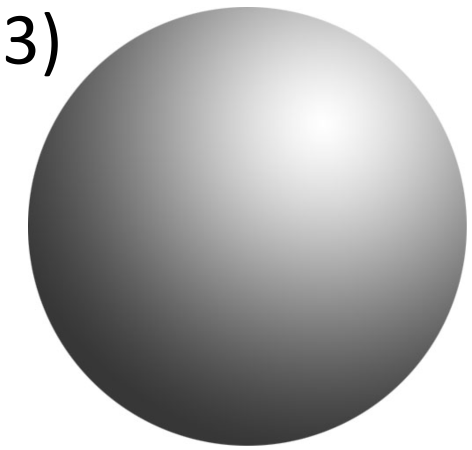
SO(2)



1-manifold

$\mathbb{R}^{3x3}$

SO(3)



3-manifold

$T_{SO(2)}$ vector space with 1 dimension

subspace of $\mathbb{R}^{2x2}$

$T_{SO(3)}$ vector space with 3 dimensions

subspace of $\mathbb{R}^{3x3}$

They are matrices

Skew-Symmetric Matrix

M is skew-symmetric matrix iff $M^T = -M$

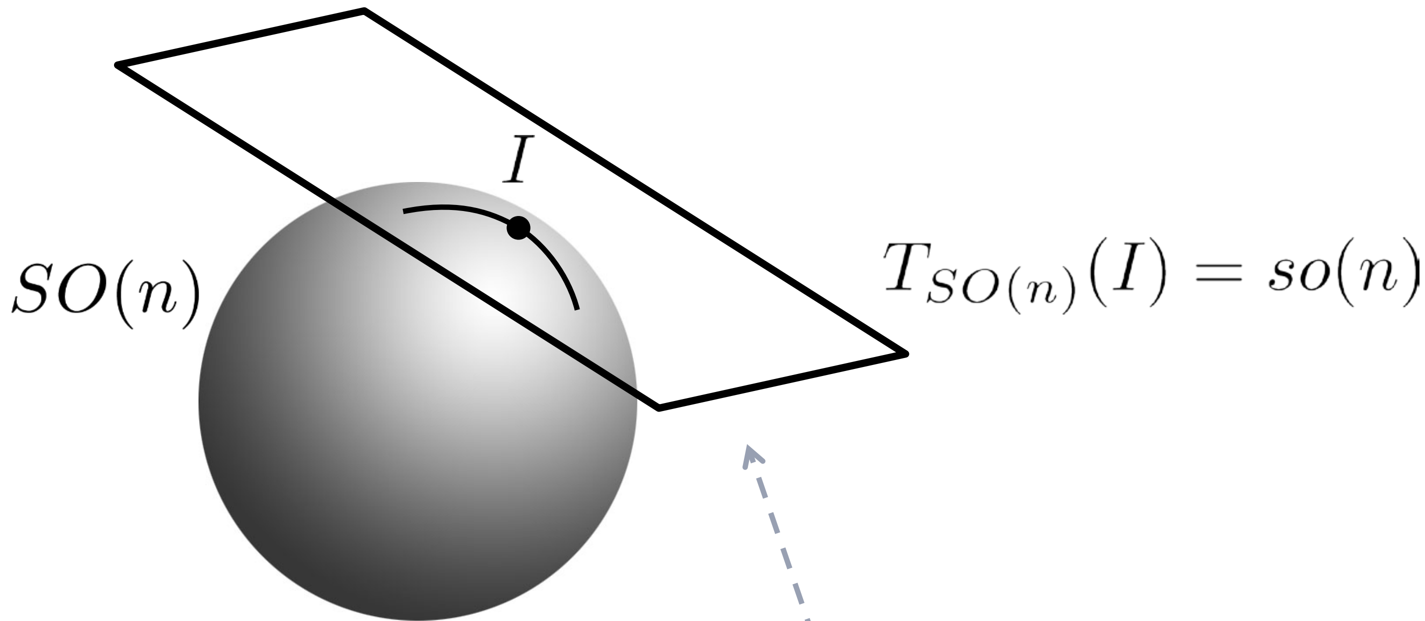
$$\begin{bmatrix} 0 & 3 & 6 \\ -3 & 0 & -1 \\ -6 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$$

$$so\left(n\right) = \left(\left\{M \in \mathbb{R}^{nxn} \mid M^T = -M\right\}, +, \cdot_e, [\,]\right)$$

Special orthogonal Lie algebra
(vector space with Lie brackets)

$$[A, B] = AB - BA$$

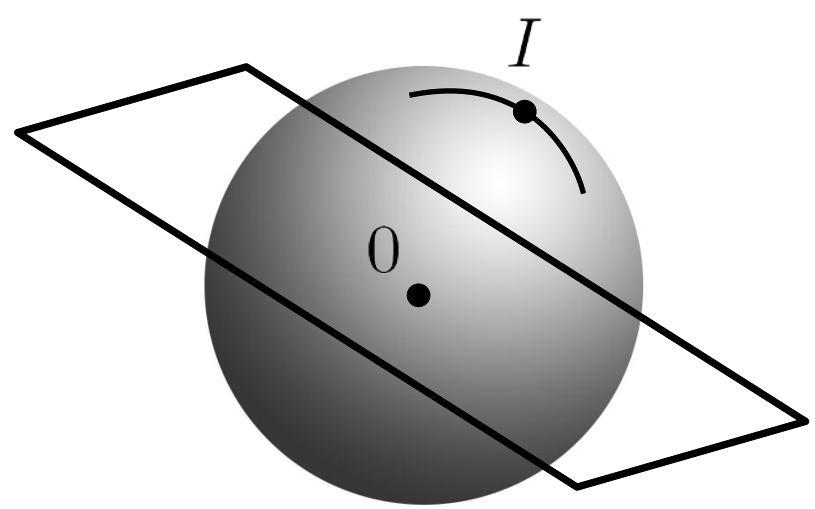
Skew-Symmetric Matrix



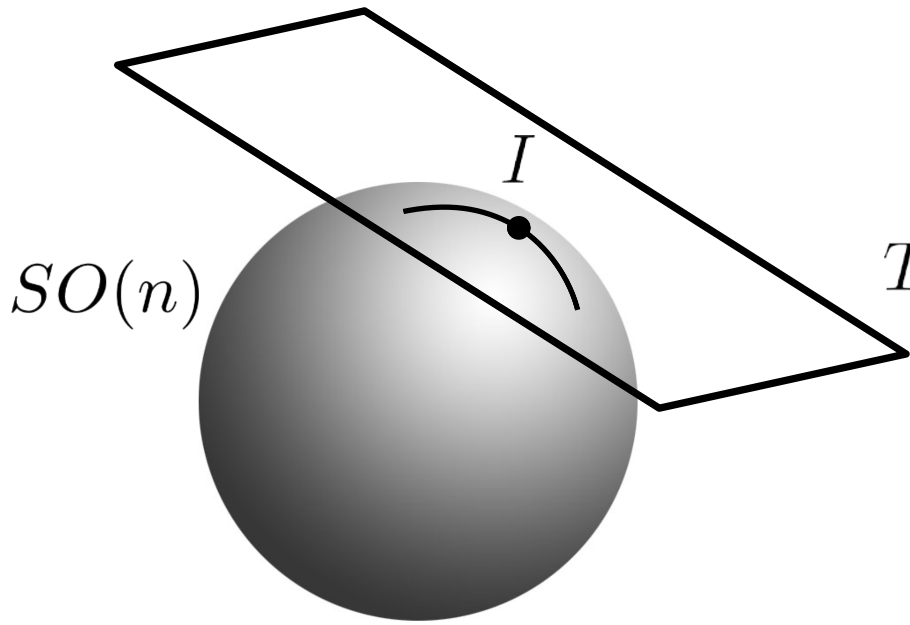
The Special orthogonal Lie algebra is the tangent space of $SO(n)$ at the identity

$so(n)$ is a vector space so it passes through the null matrix

so in reality



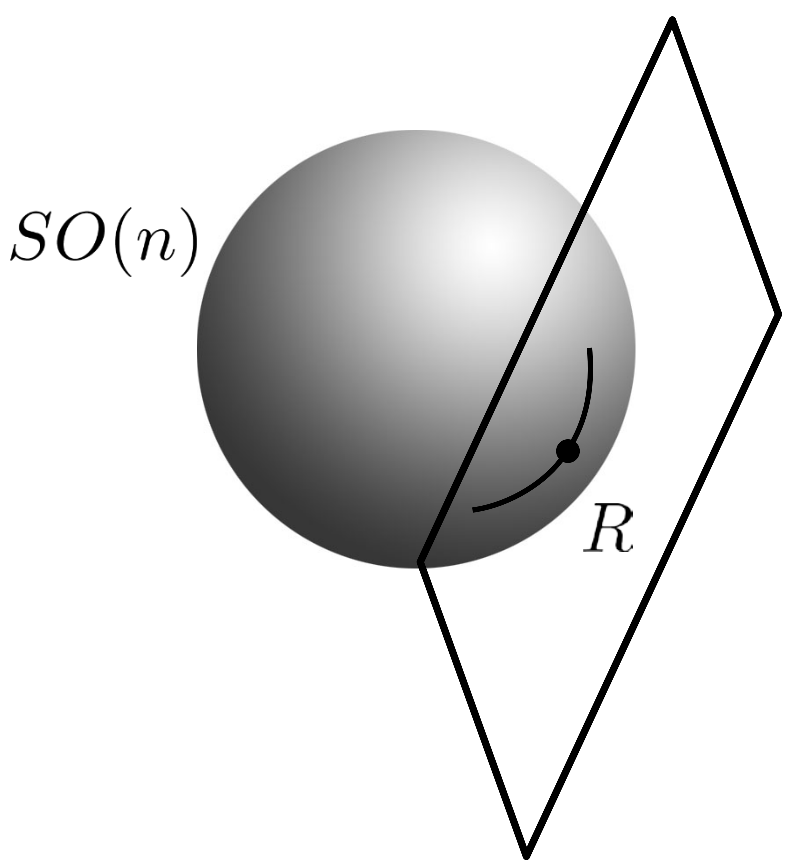
Skew-Symmetric Matrix



$T_{SO(n)}(I) = so(n)$

The Special orthogonal Lie algebra is the tangent space of $SO(n)$ at the identity

valid only at the identity



$T_{SO(n)}(R) = R \times so(n)$

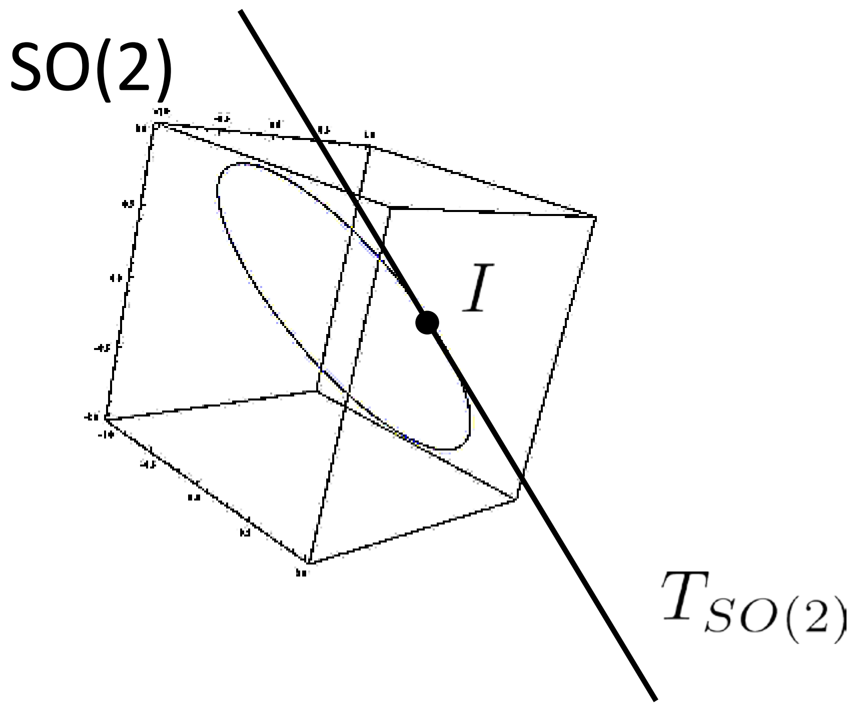
The tangent space of $SO(n)$ in any other point R is a rotated version of $so(n)$

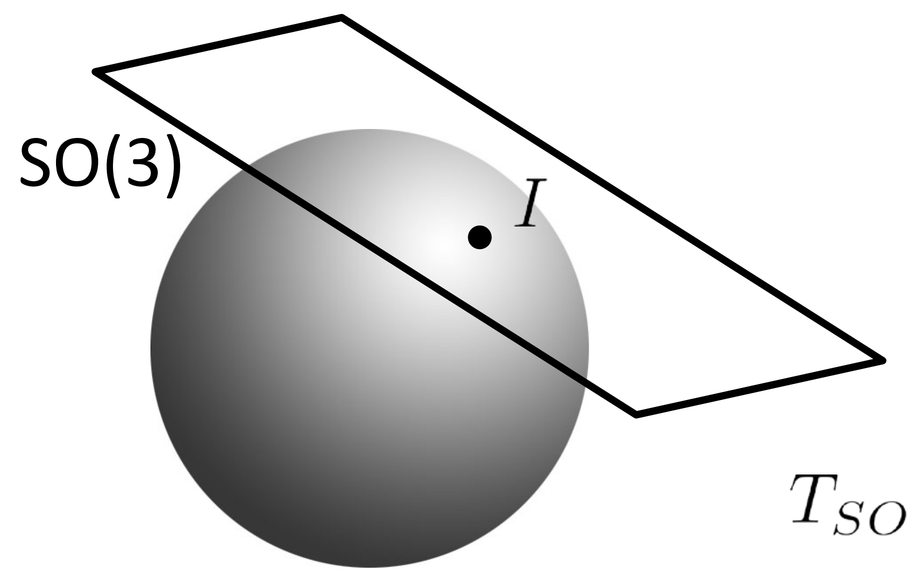
they are no more skew-symmetric matrices but rotations of them

so(2) and so(3)

$$\begin{bmatrix} 0 & 3 & 6 \\ -3 & 0 & -1 \\ -6 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$$

- $so(3)$ is a vector space of dimension 3
- $so(2)$ is a vector space of dimension 1





$T_{SO(3)}$ is a volume

an element in so(3) or so(2) represents an infinitesimal rotation from the identity matrix

The hat operator

- The hat operator in $so(3)$

$$\hat{\cdot} : \mathbb{R}^3 \to so(3)$$

$$\widehat{(x, y, z)} \to \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

- it is an isomorphism from $so(3)$ to $\mathbb{R}^3$
  (it maps + into +)

- The hat operator in $so(2)$

$$\hat{\cdot} : \mathbb{R} \to so(2)$$

$$\hat{x} \to \begin{bmatrix} 0 & -x \\ x & 0 \end{bmatrix}$$

The hat operator

- The hat operator is used to define cross-product in matrix form:

$$a \times b = \widehat{a} b \qquad\qquad \forall a, b \in \mathbb{R}^3$$

- The hat operator maps cross products into [.,.]

$$\widehat{a \times b} = \left[\widehat{a}, \widehat{b} \right]$$

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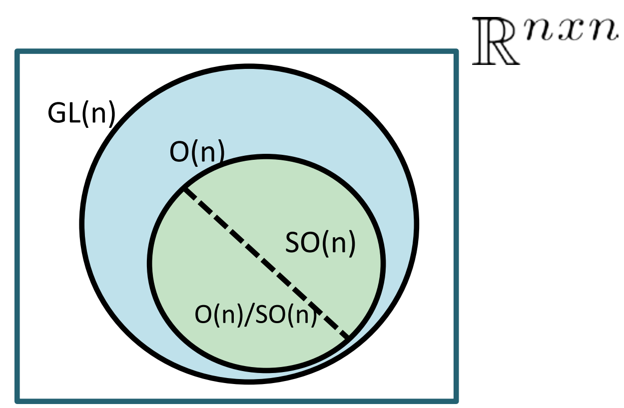
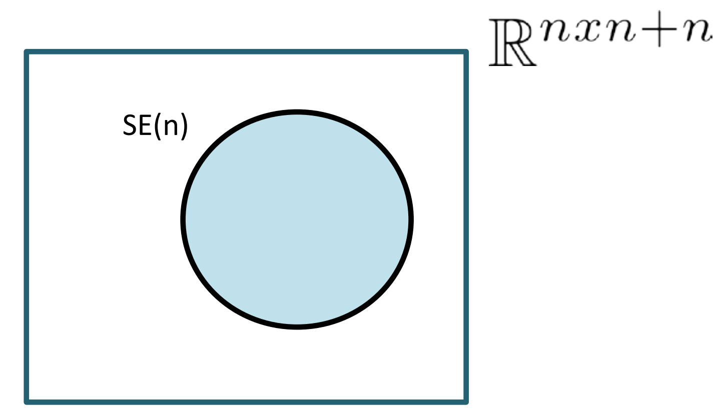
Lie Groups

- GL(n), O(n), SO(n) and SE(n) are all **Lie groups**
  (groups which are also smooth manifold where the operation is a differentiable function between manifolds)

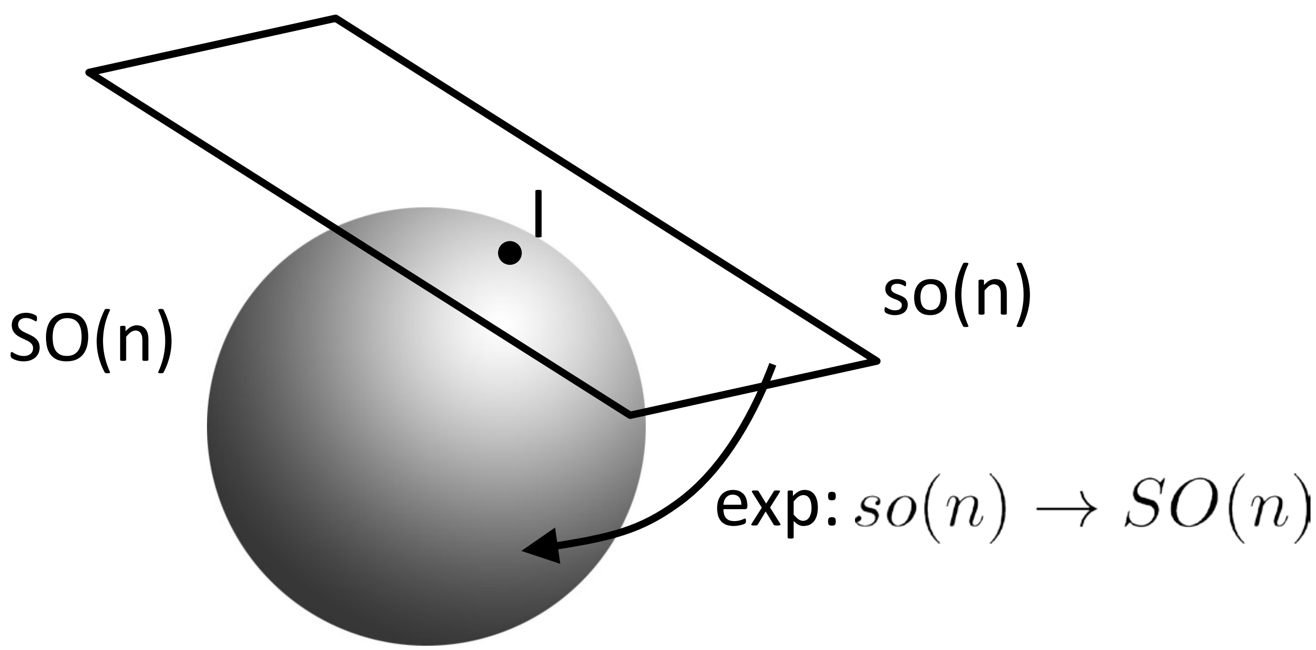




Exponential Map

- Given a Lie group **G**, with its related Lie Algebra **g = TG(I)**, there always exists a smooth map from Lie Algebra **g** to the Lie group **G** called **exponential map**

$$\exp : g \to G$$



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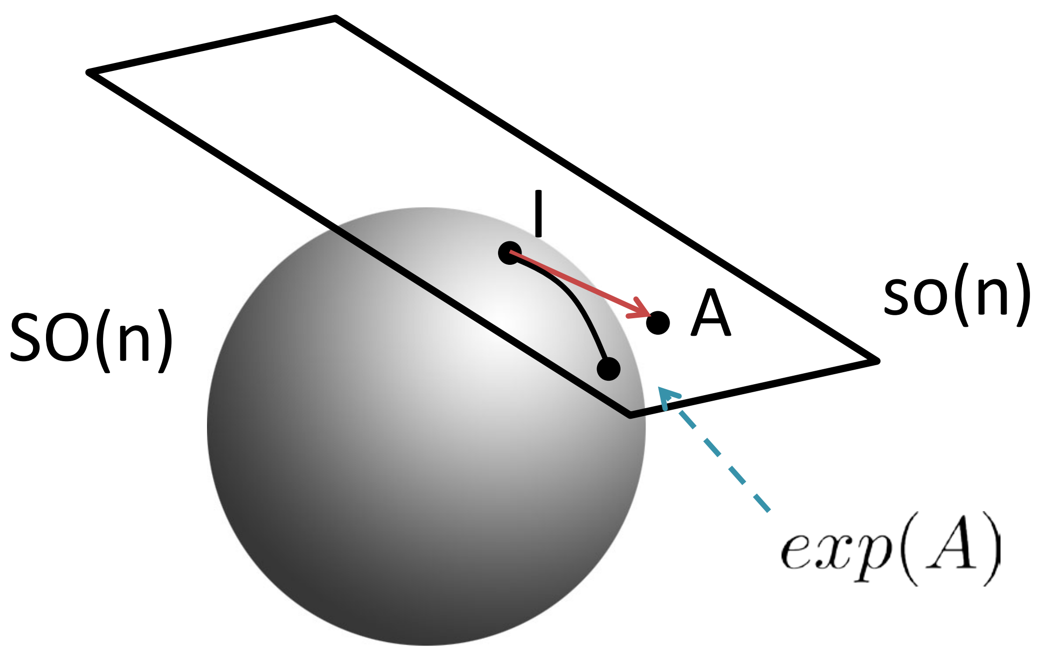


$exp(A) \;=$ **is the point in G that can be reached by traveling along the geodesic passing through the identity I in direction A, for a unit of time**

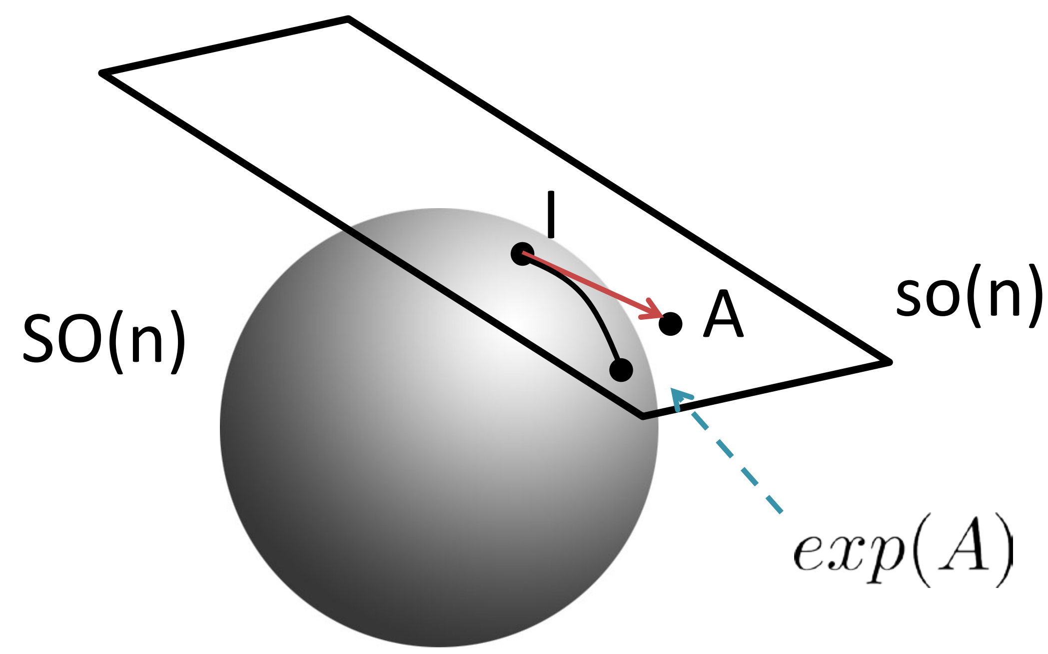
(Note: A defines also the traveling speed)

Exponential Map

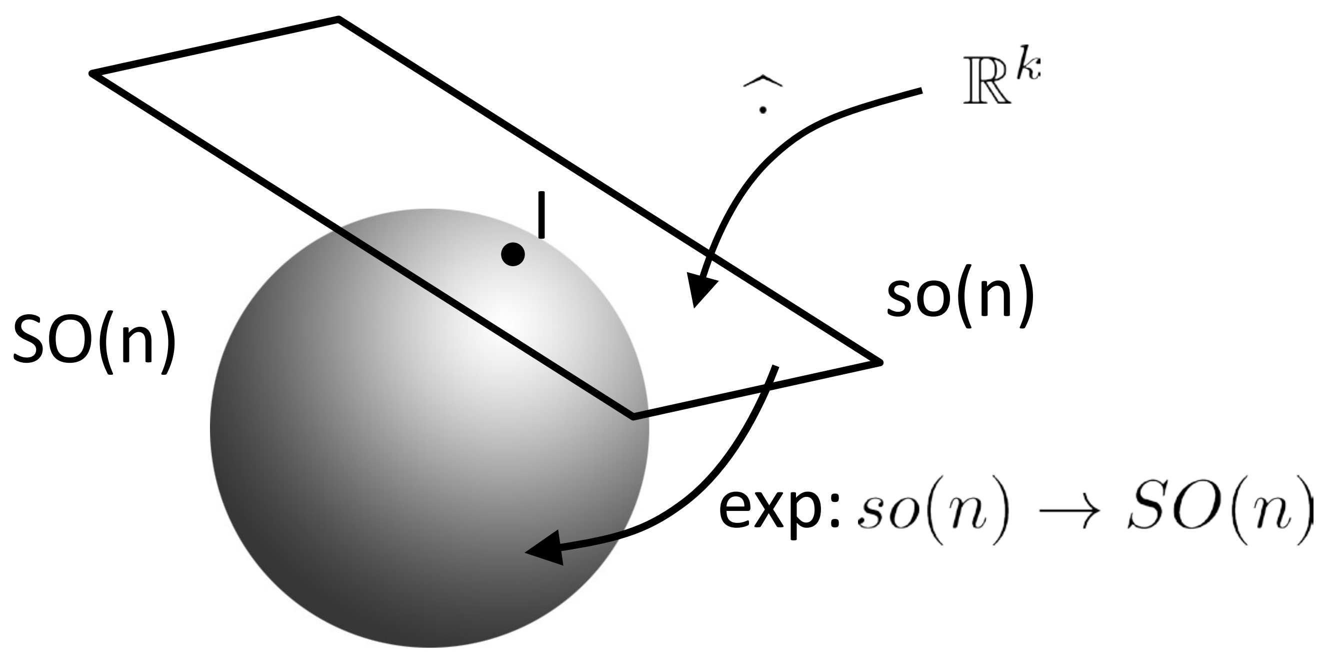
- The exponential map for a any matrix Lie group (GL(n), O(n), and SO(n)) coincides with the matrix exponential:

$$\exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

- it is a **smooth map**
- it is **surjective** (it covers the Lie Group entirely)
- it is **not injective** (is a many to one map)



Exponential Map and Hat Operator



$$\omega \in \mathbb{R}^k$$

$$\widehat{\omega} \in so(n)$$

$$\exp(\widehat{\omega}) = \sum_{k=0}^{\infty} \frac{1}{k!} \widehat{\omega}^k \quad \in SO(n)$$

- composed with the hat operator, it is a **smooth** and **surjective** map from $\mathbb{R}^k$ to SO(n) (k = the dimension of the tangent space)

$\omega \in \mathbb{R}^k \rightarrow \exp(\widehat{\omega})$ **Angle-Axis representation**

*

Properties

$$\exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

$$e^{\widehat{0}} = e^{0} = I$$ **Identity**

$$e^{-X} = (e^{X})^{-1}$$ **Inverse** $\longrightarrow$ $e^{\widehat{-\omega}} = e^{-\widehat{\omega}} = (e^{\widehat{\omega}})^{-1}$

$$e^{X+Y} \neq e^{X} e^{Y}$$ **in general not "Linear"** **(different from the standard exp in $\mathbb{R}$)**

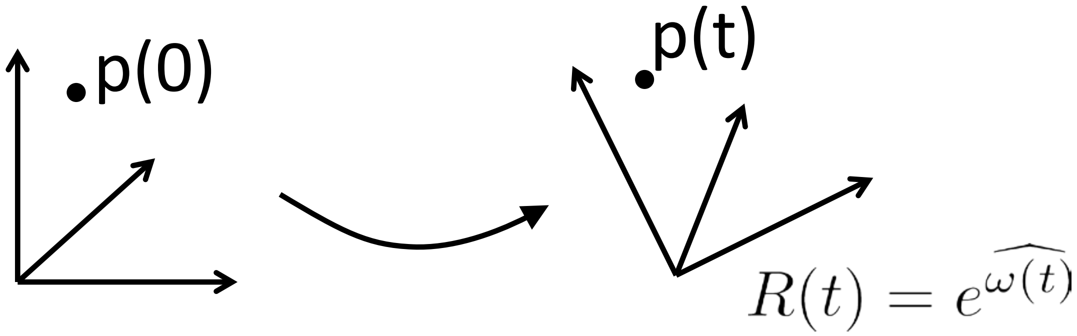
$$e^{X} e^{Y} \neq e^{Y} e^{X}$$

$$e^{sX+tX} = e^{sX} e^{tX}$$ $\forall t, s \in \mathbb{R}$

$$\partial e^{X} = \partial X e^{X} = e^{X} \partial X$$ **Derivative**

*

Physical meaning of so(3)



$$p(t) = e^{\widehat{\omega(t)}} p(0)$$ **Position**

$$\frac{\partial p}{\partial t}(t) = \frac{\partial \widehat{\omega}}{\partial t}(t) \boxed{e^{\widehat{\omega(t)}} p(0)}$$ **Velocity**

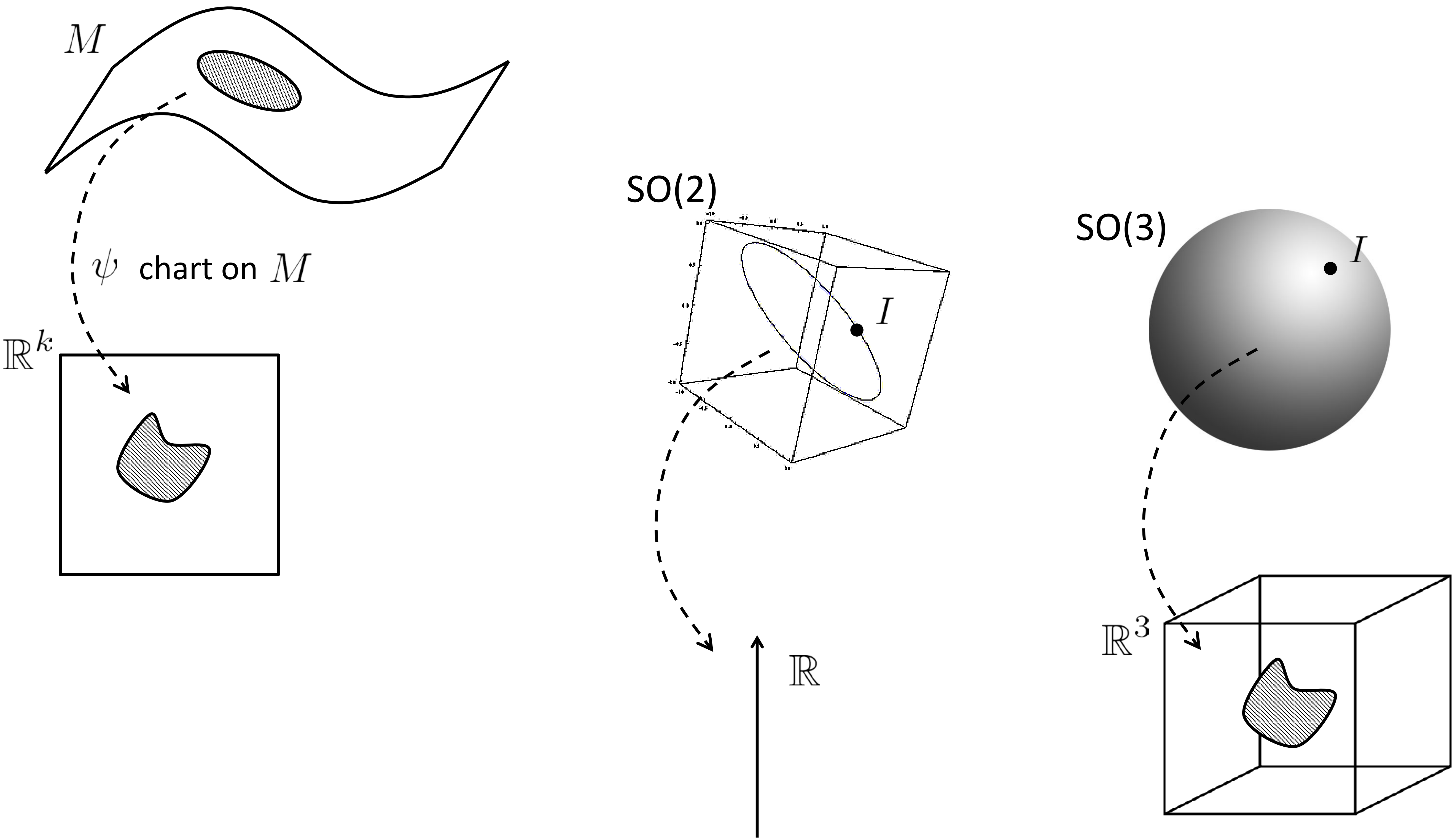
$$= \frac{\partial \widehat{\omega}}{\partial t}(t) \; p(t)$$

Spatial (angular) velocity $\in so(3)$
(transform each point in $\mathbb{R}^3$ into the corresponding speed that that point undergoes during the rotation at time t)

Content

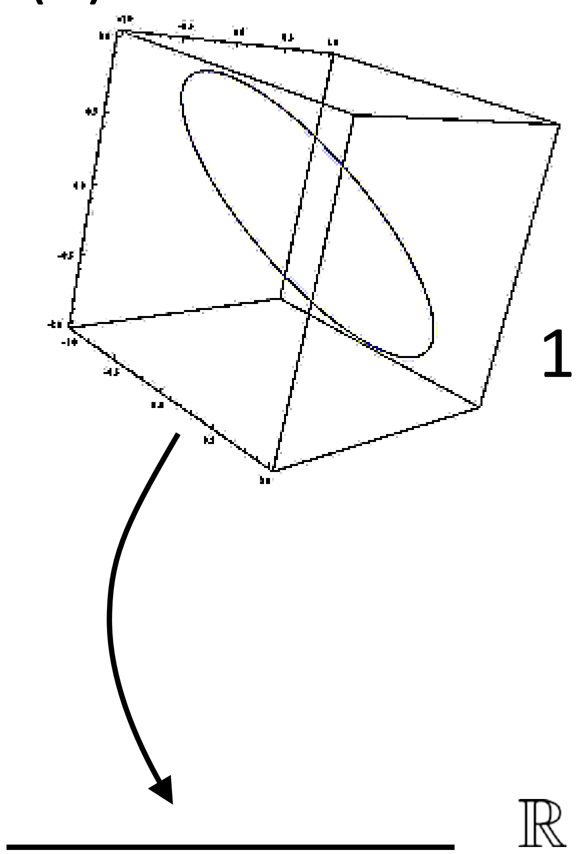
- Rigid transformations
- Matrix Groups
- Manifolds
- Lie Groups/Lie Algebras
- **Charts on SO(2) and SO(3)**

Charts on SO(2) and SO(3)



Charts on SO(2)

SO(2)



1-manifold

$\mathbb{R}$

$$\gamma : [0, 2\pi) \to SO(2)$$

$$\gamma(\theta) = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \in SO(2)$$

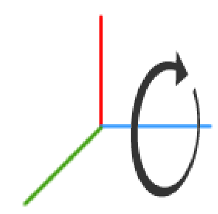
- Chart of SO(2), that cover the entire SO(2) using a single parameter

$$\begin{aligned} &\min f(T, \mathcal{O}) \\ &T \in SO(2) \end{aligned} \quad \Rightarrow \quad \begin{aligned} &\min f(\gamma(\theta), \mathcal{O}) \\ &\theta \in [0, 2\pi) \end{aligned}$$

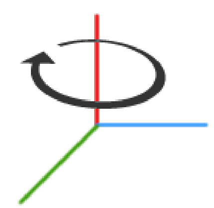
Charts on SO(3)

- **Euler's Theorem for rotations**: Any element in SO(3) can be described as a sequence of three rotations around the canonical axes, where no successive rotations are about the same axis.

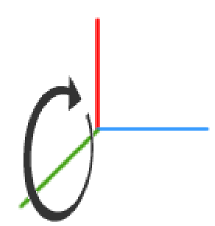
$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\alpha) & -sin(\alpha) \\ 0 & sin(\alpha) & cos(\alpha) \end{bmatrix}$$



$$R_y(\beta) = \begin{bmatrix} cos(\beta) & 0 & sin(\beta) \\ 0 & 1 & 0 \\ -sin(\beta) & 0 & cos(\beta) \end{bmatrix}$$



$$R_z(\gamma) = \begin{bmatrix} cos(\gamma) & -sin(\gamma) & 0 \\ sin(\gamma) & cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$\in SO(3)$

- For any $R \in SO(3)$ there $\exists \alpha, \beta, \gamma \mid R = R_x(\alpha)R_y(\beta)R_z(\gamma)$

- α, β, γ are called **Euler angles** of R according to the XYZ representation

SO(3): Euler angles

- Given M there are 12 possible ways to represent it

$$M \in SO(3) \iff \exists \alpha, \beta, \gamma \mid M = R_x(\alpha) R_y(\beta) R_z(\gamma)$$ XYZ

$$M \in SO(3) \iff \exists \alpha, \beta, \gamma \mid M = R_x(\alpha) R_z(\beta) R_y(\gamma)$$ XZY

$$M \in SO(3) \iff \exists \alpha, \beta, \gamma \mid M = R_x(\alpha) R_z(\beta) R_x(\gamma)$$ XZX

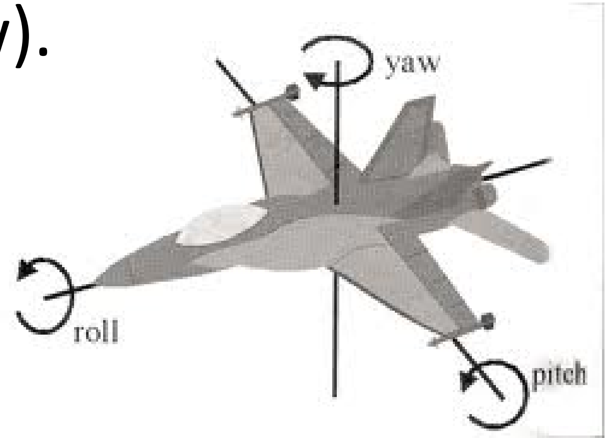
$$M \in SO(3) \iff \exists \alpha, \beta, \gamma \mid M = R_z(\alpha) R_x(\beta) R_z(\gamma)$$ ZXZ

....

Remarks: multiplication is not commutative

- Unfortunately, all of them have the same drawbacks!! (see later)

- A common representation is ZYX corresponding to a rotation first around the x-axis (roll), then the y-axis (pitch) and finally around the z-axis (yaw).



SO(3): Euler angles

- The parameterization is non-linear
- The parameterization is modular $R_x(\alpha + 2k\pi) = R_x(\alpha)$
  (but this is something that we need to live with in any representation of SO(3))
- Beside the modularity, the parameterization is not unique:

for some R in SO(3), $\exists \alpha_1, \beta_1, \gamma_1$ and $\alpha_2, \beta_2, \gamma_2$ such that

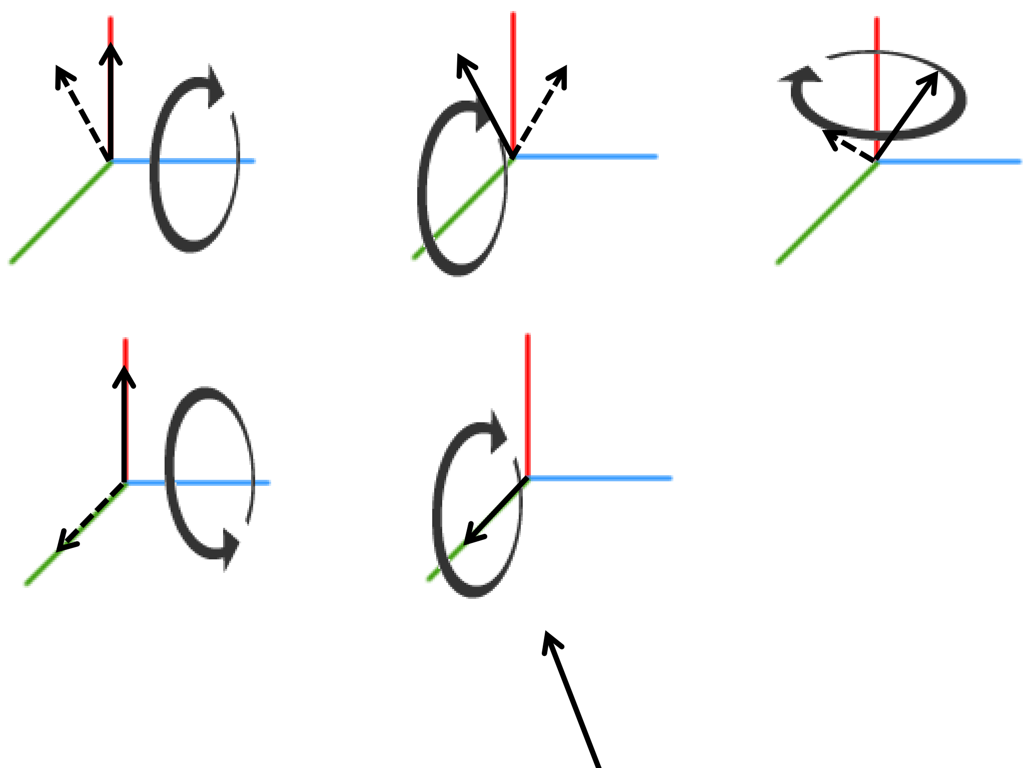
$$M = R_x(\alpha_1)R_y(\beta_1)R_z(\gamma_1)$$
$$M = R_x(\alpha_2)R_y(\beta_2)R_z(\gamma_3)$$

SO(3): Euler angles

- The parameterization have some singularities, called **gimbal lock**
- a gimbal lock happens when after a rotation around an axis, two axes align, resulting in a loss of one degree of freedom

(a)

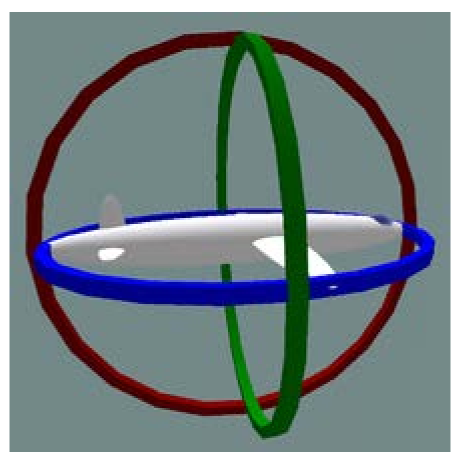
(b)



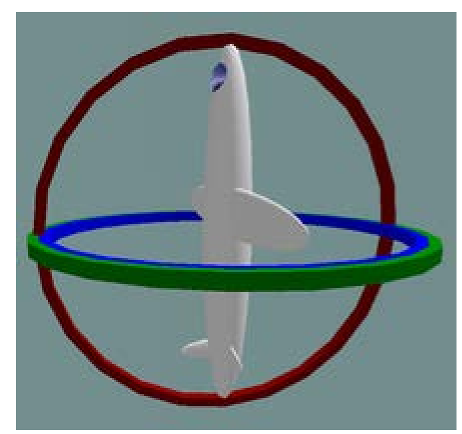
cannot move (lock)

SO(3): Euler angles

- The parameterization have some singularities, called g**imbal lock**
- a gimbal lock happens when after a rotation around an axis, two axes align, resulting in a loss of one degree of freedom
- The name gimbal lock derives from the gimbal



normal



lock [wikipedia]

- Even the most advanced modeling software uses Euler angles to parameterize the orientation of the rendering window. This is because Euler angles are more intuitive to the user. As a drawback, the gimbal lock is often noticeable.

Euler Angles and Angle/Axis

- The Euler angle representation say

$$R \in SO(3) \iff \exists \alpha, \beta, \gamma \mid R = R_x(\alpha) R_y(\beta) R_z(\gamma)$$

XYZ representation

$$= e^{\alpha \widehat{x}} e^{\beta \widehat{y}} e^{\gamma \widehat{z}}$$

possible Gimbal lock

$$\neq e^{\alpha \widehat{x} + \beta \widehat{y} + \gamma \widehat{z}}$$

no Gimbal lock

- while Euler angle define 3 rotation matrices, the angle/axis representation define a single rotation matrix identified by an element of R3