Pushforward Operator: Derivatives on a Manifold

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Dual Space

For every vector space V over $\Bbbk$, V^* is the dual space of V iff

$$V^* = L(V, \Bbbk)$$

i.e., V^* is the set of all linear maps from V to $\Bbbk$ with the inherited operation $+$ and $\cdot_e$ that makes it a vector space. The standard basis for V^* is the set of elements $\{\partial x_i\}_i$ defined as

$$\begin{array}{rccc} \partial x_i : & V & \longrightarrow & \Bbbk \\ & x & \longrightarrow & x_i \end{array}$$

where x_i is the i-th component of x. V^* is the set of the 1-forms in V.

Derivative of a Map

Given ψ a smooth map between two manifolds M and N, the derivative of ψ is defined as the function

$$d\psi : M \longrightarrow \mathbb{F}(TM, TN)$$

such that

$$d\psi(p) : T_pM \longrightarrow T_{\psi(p)}N$$

and, $\forall \gamma \in T_pM$,

$$d\psi(p)(\gamma) = [\psi \circ \gamma]_{T_{\psi(p)}N}$$

$d\psi(p)$ maps curves of the tangent space of M in p, to curves in the tangent space of N in $\psi(p)$.
$[\cdot]_{T_{\psi(p)}N}$ represents the equivalence class in the tangent space $T_{\psi(p)}N$.
$d\psi(p)$ is a linear map between tangent spaces.
The derivative operator $d(\cdot)$, also known as push-forward operator, maps

$$d : \mathbb{F}(M, N) \longrightarrow \mathbb{F}(M, \mathbb{F}(TM, TN))$$

The concept of linearity for the push-forward operator exists only for specific pairs of manifolds M, N, where i.e. the concept of vector space has sense.
In case of M and N equal to $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, the push-forward $d(\cdot)$ is equivalent to the Jacobian operator $J(\cdot)$ which produces a linear approximation of the input function around a specific point in its domain.

Derivative of a Smooth Curve

A smooth curve $\gamma : \mathbb{R} \longrightarrow M$ over a manifold M can be viewed as a smooth map from the manifold $\mathbb{R}$ to the manifold M. The concept of derivative can be inherited as

$$d\gamma\left(t\right) : T_t\mathbb{R} \longrightarrow T_{\gamma(t)}M$$

Since $T_t\mathbb{R}$ contains one direction but infinite speeds, and due to the linearity of $d\gamma\left(t\right)$, $d\gamma\left(t\right)$ can be represented with a unique element in TM, i.e. the curve corresponding to [1], i.e. the class of curves passing though t with derivative equal to 1.
We can therefore define

$$\begin{array}{rccc} \partial\gamma : & \mathbb{R} & \longrightarrow & TM \\ & t & \longrightarrow & d\gamma\left(t\right)\left(\left[1\right]\right) \end{array}$$

$\partial\gamma\left(t\right)$ is therefore the class of curve in $T_{\gamma(t)}M$ with the same tangent and speed of γ in t.
Remember that, the tangent space of a manifold in a point p is the set of all the classes of curves each passing through p, such that curves having the same tangent and speed in p are grouped in the same class.

Derivative of a Functional

A smooth functional over M, $\psi : M \longrightarrow \mathbb{R}$ can be viewed as a smooth map between M and the manifold $\mathbb{R}$. The concept of derivative is inherited as

$$d\psi : M \longrightarrow \mathbb{F}\left(TM, T\mathbb{R}\right)$$

and, since $T\mathbb{R} = \mathbb{R}$,

$$d\psi\left(p\right) : T_pM \longrightarrow \mathbb{R}$$

$d\psi\left(p\right)$ belongs therefore to the dual of T_pM, i.e.

$$d\psi : M \longrightarrow T^*M$$

where T^*M is the bundle comprising of all $\left(T_pM\right)^*$ for every $p \in M$.
From the initial definition of derivative, we have that $\forall\gamma \in T_pM$,

$$d\psi\left(p\right)\left(\gamma\right) = \psi \circ \gamma$$

where the class operator $\left[\cdot\right]$ as we are operating in $\mathbb{R}$.

Cotangent Space

The cotangent space of M in p is defined as the dual of the tangent space of M in p

$$T_p^*M = \left(T_pM\right)^*$$

Note that, this definition always exists as T_pM is always a vector space even if it is made of curves. T_p^*M maps curves to real numbers.
From a derivative point of view, the tangent bundle of a manifold TM consists of all the possible derivatives of smooth curves over M, i.e. of maps of kind $\gamma : \mathbb{R} \longrightarrow M$. The cotangent bundle T^*M instead consists of all the possible derivatives of smooth funcionals on M, i.e. of maps of kind $\gamma : M \longrightarrow \mathbb{R}$.

Derivative for Manifolds immerse in Vector Spaces

The concept of derivative allows us to bundle a vector space to each point on the manifold such that it is isomorphic to the tangent space and has a meaning in terms of derivative. Once this space is bundled we can use, as tangent space, either the original one (made with curves), or a second one (made with elements of the vector space in which the manifold is immersed in).
This creates the duality visible in $SO(3)$ of an element of a tangent space being both a curve and an anti-simmetric matrix.

Derivative of a Smooth Curve

If M is a manifold immerse in a vector space V, a smooth curve over M is also a smooth curve over V. If V has its own concept of derivative, then there exists a second concept of derivative for this curve inherited from this space, which is

$$\partial^V \gamma(t) = \lim_{\varepsilon \longrightarrow 0} \frac{\gamma(t+\varepsilon) - \gamma(t)}{\varepsilon}$$

In these scenarios, it is common to connect, when possible, these two concepts of derivatives by an isomorphism. This connection does not always exists but when it does it forces a unique isomorphism between a subspace of V, $S_p \subseteq V$ and T_pM for every $p \in M$.

$$S_p \longleftrightarrow T_pM$$

For every curve γ such that $\gamma(t) = p$, the isomorphism $\Omega_p : T_pM \longrightarrow S_p$ follows

$$\Omega_p([\partial\gamma(t)]) = \partial^V \gamma(t)$$

i.e. $\Omega_p(x)$ is the element of S_p that results from deriving at a specific time t any smooth curve that passes through p at time t and has as pushforward ∂ the element x.
SO(3) and SE(3): $SO(3)$ is immerse in $V = \mathbb{R}^{3x3}$, the subspace S_R for $R \in SO(3)$ is defined as

$$S_R = \left\{\omega R \mid \forall\omega \in \mathbb{R}^{3x3} \text{ anti-symmetric}\right\}$$

This can be proven by deriving a curve over $SO(3)$ around R which leads to an anti-symmetric matrix multiplied to R.
S_I, i.e. the tangent space at the identity is defined as

$$S_I = \left\{\omega \mid \forall\omega \in \mathbb{R}^{3x3} \text{ anti-symmetric}\right\}$$

and it is isomorphic to the Lie Algebra $so(3)$ using the same isomorphism.

Derivative of a Functional

As for the smooth curves, the concept of derivative for functionals over M generates an isomorphism between V^* and T_p^*M. Specifically from $S_p^* \longleftrightarrow T_p^*M$ which is the same S_p^* found for the smooth curves. This is much easier to prove due to the definition of $d\psi : M \longrightarrow T^*M$ where it clearly make a reference to $T_pM \longrightarrow \mathbb{R}$. So if we admit that there exists a connection between $S_p \longleftrightarrow T_pM$ then there should be a connection $S_p^* \longleftrightarrow T_p^*M$.
SO(3) and SE(3): S_R^* for $R \in SO(3)$ is defined as

$$S_R^* = L(S_R, \mathbb{R})$$

i.e. as the set of all the linear maps from the $3x3$ matrices S_R and $\mathbb{R}$. Since they are linear maps, each of its elements $v \in S_R^*$ can be represented as

$$v(m) = km^{\downarrow}$$

where m is a $3x3$ matrix in S_R, while the vector $k \in \mathbb{R}^9$ uniquiely identify the element v in S_R^*.

PullBack of a Map

Given ψ a smooth map between two manifolds M and N, elements in M transform to N with ψ, elements in TM transform with $d\psi$, while elements in T^*M transform with $d^*\psi$.

$$\begin{array}{rcccc} \psi: & M & \longrightarrow & N & \\ d\psi: & M & \longrightarrow & TM & \longrightarrow & TN \\ d^*\psi: & M & \longrightarrow & T^*N & \longrightarrow & T^*M \end{array}$$

$d\psi$ is called **push-forward** of ψ**.**
$d^*\psi$ is called **pull-back** of ψ, and defined for every $p \in M$ as

$$d^*\psi\,(p)\,(\gamma) = \gamma \circ d\psi\,(p)$$

where $\gamma : TN \longrightarrow \mathbb{R}$ is a smooth functional in TN, and $\gamma \circ d\psi\,(p)$ is a map from $TM \longrightarrow TN \longrightarrow \mathbb{R}$, therefore $d^*\psi\,(p)\,(\gamma) \in T^*M$.

Transformations: Covariant and Contravariant

Since elements in TM and elements in T^*M transforms in different ways with a change of reference system, they are called with different names, precisely, **vectors** and **co-vectors**, respectively.
Since the vectors transform in a different way that the point on a manifold, they are called **contra-variant** (they varies in a different way).
Since the co-vectors transform in a similar way that the point on a manifold, they are called **covariant**.