Jacobian of a homography

The homography f associated with the 3×3 matrix M is a function $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined as

$$f\left(x,y\right) \sim M \times \left(x,y,1\right)^T$$

The Jacobian of f is

$$\begin{aligned} Jf\left(x,y\right) &= \frac{1}{M_{[2,:]}\cdot\left(x,y,1\right)} M_{[0:2,0:2]} - \frac{1}{\left(M_{[2,:]}\cdot\left(x,y,1\right)\right)^2}\left(M_{[0:2,:]}\times\left(x,y,1\right)^T\right)\times M_{[2,0:2]} \\ &= \frac{1}{h\left(x,y\right)^2}\begin{pmatrix} M_{0,0}h\left(x,y\right) - g^0\left(x,y\right)M_{2,0}\ , & M_{0,1}h\left(x,y\right) - g^0\left(x,y\right)M_{2,1} \\ M_{1,0}h\left(x,y\right) - g^1\left(x,y\right)M_{2,0}\ , & M_{1,1}h\left(x,y\right) - g^1\left(x,y\right)M_{2,1} \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} h\left(x,y\right) &= M_{[2,:]}\cdot\left(x,y,1\right) = M_{2,0}\ x + M_{2,1}\ y + M_{2,2} \\ g^0\left(x,y\right) &= M_{[0,:]}\cdot\left(x,y,1\right) = M_{0,0}\ x + M_{0,1}\ y + M_{0,2} \\ g^1\left(x,y\right) &= M_{[1,:]}\cdot\left(x,y,1\right) = M_{1,0}\ x + M_{1,1}\ y + M_{1,2} \end{aligned}$$

Proof:

$$\begin{aligned} f^0\left(x,y\right) &= \frac{g^0\left(x,y\right)}{h\left(x,y\right)} \\ f^1\left(x,y\right) &= \frac{g^1\left(x,y\right)}{h\left(x,y\right)} \end{aligned}$$

$$\begin{aligned} \frac{df^i\left(x,y\right)}{dx} &= \frac{g^i_x\left(x,y\right)}{h\left(x,y\right)} - \frac{g^i\left(x,y\right)h_x\left(x,y\right)}{h\left(x,y\right)^2} \\ &= \frac{g^i_x\left(x,y\right)h\left(x,y\right) - g^i\left(x,y\right)h_x\left(x,y\right)}{h\left(x,y\right)^2} \\ \frac{df^i\left(x,y\right)}{dy} &= \frac{g^i_y\left(x,y\right)h\left(x,y\right) - g^i\left(x,y\right)h_y\left(x,y\right)}{h\left(x,y\right)^2} \end{aligned}$$

where

$$\begin{aligned} g^i_x\left(x,y\right) &= M_{i,0} \\ g^i_y\left(x,y\right) &= M_{i,1} \\ h_x\left(x,y\right) &= M_{2,0} \\ h_y\left(x,y\right) &= M_{2,1} \end{aligned}$$

Note that $Jf : \mathbb{R}^2 \longrightarrow \mathbb{R}^{2\times 2}$ and $Jf\left(x,y\right) = \left[f_x\left(x,y\right), f_y\left(x,y\right)\right]$.